Examples of chromatic redshift in algebraic K-theory
THH and trace methods seminar - Talk 17

Maite Carli

March 2025

Contents

1 Introduction

In this talk, we address the following question: if E is an $\mathbb{E}_\infty$-ring spectrum of height n, then what is the height of the algebraic K-theory of E, $K(E)$? We do so following Allen Yuan's arguments in [13].

Throughout let p be a fixed prime and work p-completed where necessary.

Definition 1.1. An $\mathbb{E}_\infty$-ring spectrum E is of *height* n if $T(n) \otimes E \not\simeq 0$ but $T(n+1) \otimes E \simeq 0$.

Remark 1.2. In this case, by a theorem of Hahn (Thm. 1.1, [5]), it holds that

$$T(m) \otimes E \not\simeq 0 \ \forall 0 \leq m \leq n \text{ and } T(m) \otimes E \simeq 0 \ \forall m > n.$$

Some historical remarks about redshift:

- In 2002, Ausoni and Rognes observed, while doing calculations for $K(ku)$, that K-theory has a tendency to shift height up by one. This idea became known as *Redshift philosophy.*
- Various computations by Blumberg, Mandell, Ausoni and Rognes among others kept confirming this phenomenon.
- In 2020, Clausen Matthew Naumann Noel showed that K-theory cannot shift the height by more than one:

  Theorem 1.3 ([4, Thm. A])**.** *Let R be an $\mathbb{E}_\infty$-ring and $n \geq 0$. If $L_{T(n)}R = 0$, then $L_{T(n+1)}K(R) = 0$.*

- In this talk, we will focus on redshift for Lubin-Tate theories and for iterated K-theory of fields. These results were shown by Yuan in 2021:

Theorem 1.4 ([13, Thm. A])**.** *Let k be a perfect field of characteristic p, let $\mathbb{G}_0$ be a 1-dimensional formal group over k of height $n \geq 1$, and let $E_n = E_{k,\mathbb{G}_0}$ denote the associated Lubin-Tate theory. Then, $L_{T(n+1)}K(E_n) \not\simeq 0$.*

Theorem 1.5 ([13, Thm. B])**.** *Let $n \geq 0$ and let $K^{(n)}$ denote the n-fold iterate of algebraic K-theory. Then, for any field k of characteristic different from p, the spectrum $L_{T(n)}K^{(n)}(k)$ is nonzero.*

- The redshift conjecture was finally shown to be true in 2022 by Burklund-Schlank-Yuan in [3]. The proof relies on theorem 1.4, the fact that there cannot exist a ring map from the zero ring into a non-trivial ring and the new ingredient that any non-trivial, $T(n)$-local $\mathbb{E}_\infty$-ring admits a ring map to some Lubin-Tate theory E_n.

We briefly recall a few facts on $T(n)$ and its relation to $K(n)$:

Definition 1.6. For X a finite type n spectrum[1] and a v_n-self map f[2], define

$$T(n) = \operatorname{colim}(X \xrightarrow{f} \Sigma^{-d}X \xrightarrow{f} \dots) = X[f^{-1}].$$

Remark 1.7. By asymptotic uniqueness of v_n-self maps and the thick subcategory theorem, the Bousfield class of $T(n)$ depends only on n.

Lemma 1.8. *A homotopy commutative ring spectrum Y is $T(n)$-acyclic if and only if it is $K(n)$-acyclic.*

Proof. We begin by assuming that Y is $T(n)$-acyclic. Then, $Y \otimes T(n) \otimes K(n) \simeq 0$ as well. Observe that $T(n) \otimes K(n)$ is a $K(n)$-module and as such it is equivalent to a direct sum of shifts of $K(n)$. In other words, $K(n)$ is a retract of $T(n) \otimes K(n)$. Hence, Y must be $K(n)$-acyclic.

The converse is an application of the nilpotence theorem. Observe that if X is of type n, then $X \otimes DX$ is also of type n. The latter is always a homotopy commutative ring spectrum and we can define $T(n)$ using this spectrum, thus endowing it with a homotopy commutative ring structure. As Y was assumed to be a ring spectrum, $Y \otimes T(n)$ is too. To show that $Y \otimes T(n) \simeq 0$, it suffices to show that $1 \in \pi_*(Y \otimes T(n))$ is nilpotent. By the nilpotnece theorem, this is the case if and only if $1 \in \operatorname{Ker}(\pi_*(Y \otimes T(n) \to K(m)_*(Y \otimes T(n)))$ for all non-negative integers m. If $m = n$, this is true by assumption. Else, observe that

$$0 \simeq T(n) \otimes K(m) = \operatorname{colim}(K(m) \otimes X \otimes DX \xrightarrow{f} K(m) \otimes X \otimes DX \xrightarrow{f} \dots).$$

Indeed, as f is a v_n-self map, it is nilpotent on $K(m)$-homology and thus the colimit is trivial. □

2 Reversing blueshift ([13, Sec. 2])

To understand why K-theory shifts the height up by one, let us first analyse a case where height gets shifted down. Recall the following theorem:

Theorem 2.1 (Kuhn)**.** *Let $n \geq 0$ and let X be a $T(n)$-local spectrum with an action of C_p. Then $L_{T(n)}X^{tC_p} \simeq 0$.*

Hence, $(-)^{tC_p}$ is seen to shift the height down by one. This is called blueshift.

[1] i.e. such that $K(n)_*(X) \neq 0$, but $K(m)_*(X) = 0$ $\forall m, n$.

[2] i.e. f is an isomorphism on $K(n)$-homology and nilpotent on $K(m)$-homology for $m \neq n$.

Example 2.2. Let $X = KU_p$. This is $T(1)$-local. Endow it with the trivial C_p-action and consider X^{tC_p}. Its homotopy groups are given by

$$\pi_*(KU_p^{tC_p}) = \hat{\mathbb{Z}_p}[\beta^{\pm}]((x))/(\beta^{-1}((1+\beta x)^p - 1)) \cong \hat{\mathbb{Q}_p}(\xi_p)((x)).$$

Here, inverting x also inverted p via the relation, the p^{th}-root of unity corresponds to $1 + \beta x$. Thus, X^{tC_p} is rational, i.e. of height 0, one height less than what we started with.

Remark 2.3. To calculate the homotopy groups we used that for E complex oriented with a trivial C_p-action

$$\pi_*(E^{tC_p}) = E_*((x))/([p](x))$$

where x is in degree 2 and $[p]$ denotes the p-series of the formal group law associated to E.

If we could construct a procedure that shifts the height of X^{tC_p} back up by one, we would get an instance of redshift. Hence, we wonder: can we reverse blueshift? An answer to this will be theorem 2.8. First, we consider the following example to gain some intuition.

Example 2.4. Consider the complex-oriented spectrum $\mathbb{Z}$ endowed with the trivial C_p-action. This is of height 0. By remark 2.3, $\pi_*(\mathbb{Z}^{tC_p}) = \mathbb{F}_p((x))$, hence we think of $\mathbb{Z}^{tC_p}$ as having height -1 (it vanishes already rationally). Observe that C_p is a subgroup of the circle group S^1, which can also be seen as acting trivially on $\mathbb{Z}$. Thus, $\mathbb{Z}^{hC_p}$, $\mathbb{Z}_{hC_p}$ and $\mathbb{Z}^{tC_p}$ inherit residual $S^1/C_p \cong S^1$-actions. Consider the subgroup $C_{p^2}/C_p < S^1/C_p$ and let us analyse what effect taking its homotopy fixed points has.
Consider the map ${\mathbb{Z}^{hC_p}}^{hC_{p^2}/C_p} \to {\mathbb{Z}^{tC_p}}^{hC_{p^2}/C_p}$. The source can be identified with $\mathbb{Z}^{hC_{p^2}}$[3], whose homotopy groups are given by $\mathbb{Z}[[x]]/(p^2x)$. This knowledge enables us to compute the homotopy groups of ${\mathbb{Z}^{tC_p}}^{hC_{p^2}/C_p}$ by comparing the two homotopy fixed point spectral sequences (this is left to the reader). One finds that $\pi_*({\mathbb{Z}^{tC_p}}^{hC_{p^2}/C_p}) = \mathbb{Z}/p^2((x))$.
Thus, taking fixed points with respect to C_{p^2}/C_p has added a power of p back into $\mathbb{Z}^{tC_p}$, i.e. multiplication by p is no longer zero on the homotopy groups, only multiplication by p^2. This gives the hope that at the colimit, i.e. taking S^1/C_p-fixed points, one gets back a rational spectrum (one with p inverted + work p-completed) and one has reversed blueshift.

This hope turns out to be justified. As the following lemma states, taking S^1/C_p-fixed points is the same as applying $(-)^{tS^1}$ in the p-completed setting. Then, proposition 2.6 will show that, unlike $(-)^{tC_p}$, the functor $(-)^{tS^1}$ does not lower the height.

Lemma 2.5 (Tate-orbit)**.** *Let X be a bounded below spectrum with S^1-action. Then there is a natural map $X^{tS^1} \to {X^{tC_p}}^{hS^1/C_p}$ which exhibits the target as the p-completion of the source* [4]*.*

Proposition 2.6. *Let E be a homotopy commutative ring spectrum with S^1-action such that $L_{T(n)}E \not\simeq 0$. Then, $L_{T(n)}(E^{tS^1}) \not\simeq 0$.*

Proof. We begin by reducing to the case where E is complex oriented. By lemma 1.8, $L_{T(n)}E \simeq 0$ if and only if $L_{K(n)}E \simeq 0$. In this case, $L_{K(n)}K(n)\otimes E \simeq 0$ as well (since $E \otimes K(n) \simeq 0$). Tensoring with the unit, one obtains a ring map $E \to E \otimes K(n)$. Applying $L_{K(n)}$ to this map gives that $L_{K(n)}E \not\simeq 0$ if $L_{K(n)}E \otimes K(n) \not\simeq 0$ as there are no ring maps from 0 to a non-trivial ring. Combining these observations, we find that

$$L_{T(n)}E \not\simeq 0 \iff L_{K(n)}E \not\simeq 0 \iff L_{K(n)}E \otimes K(n) \not\simeq 0 \iff L_{T(n)}E \otimes K(n) \not\simeq 0.$$

[3]For a spectrum X with an action of a group G and N a normal subgroup, it holds that ${X^{hN}}^{hG/N} \simeq X^{hG}$.

[4]Here C_p acts on X via its induced action as subgroup of S^1, and X^{tC_p} is endowed with a residual S^1/C_p-action as in the previous example

Moreover, as $(-)^{tS^1}$ is lax symmetric monoidal, there is a ring map $E^{tS^1} \to (E \otimes K(n))^{tS^1}$. Hence, by the same argument as above, if $L_{T(n)}(E \otimes K(n))^{tS^1} \not\simeq 0$, then $L_{T(n)}E^{tS^1} \not\simeq 0$. Replacing E by $E \otimes K(n)$, we can thus without loss of generality assume that E is complex oriented.
Then, it holds that $E^{tS^1} \simeq E^{hS^1}[x^{-1}]$ (this is an enhancement of remark 2.3) and

$$E^{hS^1} = E^{\Sigma^\infty_+ \mathbb{C}P^\infty} \simeq \oplus_{n \geq 0} \Sigma^{2n} E$$

(see [8, Lecture 4, Prop. 7] for the last equivalence). In particular, E is a summand/retract of E^{tS^1} and thus $L_{T(n)}E^{tS^1}$ cannot vanish if $L_{T(n)}E$ does not. □

Remark 2.7. These last two results are where the redshift happens. Combining them, one gets that $(-)^{hS^1/C_p}$ shifts the height of e^{tC_p}, for e some connective spectrum, up by one. In more detail, by blueshift (thm. 2.1), the height of e^{tC_p} is one lower than the height of e (at least if e is $T(n)$-local). By lemma 2.5, $e^{tC_p hS^1/C_p} \simeq e_p^{tS^1}$ and, by proposition 2.6, $e_p^{tS^1}$ is of the same height as e.

The assumption that e from the previous remark is connective is crucial. Lemma 2.5 only applies to bounded below spectra and one can show that $(E^{tC_p})^{hS^1/C_p}$ might not be equivalent to $E_p^{tS^1}$ when E is not bounded below (for example for $E = KU_p$; see [13, Rmk. 2.8].)

However, exploiting these observations, we still get a general statement that "K-theory reverses blueshift":

Theorem 2.8 ([13, Thm. C])**.** *Let E be an $\mathbb{E}_\infty$-ring spectrum such that $L_{T(n)}$ is nonzero. Then $L_{T(n)}K(E^{tC_p})$ is nonzero.*

Proof. We aim to apply the previous observations on reversing blueshift. So we begin by reducing to the case where E is connective. We do this thanks to the following two results.

Theorem 2.9 ([7, Thm. A])**.** *Let $n \geq 1$, and let $A \to B$ be a map of $\mathbb{E}_1$-ring spectra which is a $T(n-1) \oplus T(n)$-equivalence. Then $K(A) \to K(B)$ is a $T(n)$-equivalence.*

Lemma 2.10 ([4, Lem. 4.7])**.** *Let X be a coconnective spectrum. Then X^{tC_p} is $T(n)$-acyclic for all $n \geq 0$.*

Roughly, to prove this lemma, one observes that $(-)^{tC_p}$ commutes with filtered colimits on coconective spectra. Hence, one can use the Whitehead tower to reduce to the case when X is concentrated in a single degree. Then, X^{tC_p} is a p-torsion $\mathbb{Z}$-module and thus $T(n)$-acyclic for all n ($T(0)$-acyclic because of p-torsion and $T(n)$-acyclic for $n \geq 1$ because $\mathbb{Z}$ is).

Combining these results, one finds that the map $K((\tau_{\geq 0}E)^{tC_p}) \to K(E^{tC_p})$ is a $T(n)$-equivalence for all $n \geq 1$: the cofiber of $(\tau_{\geq 0}E)^{tC_p} \to E^{tC_p}$ is of the form X^{tC_p} for X coconnective, thus by lemma 2.10 that map is a $T(n)$-equivalence for all n. By theorem 2.9, it remains a $T(n)$-equivalence for $n \geq 1$ when applying K. We have thus reduced to showing the result for $e \coloneqq \tau_{\geq 0}E$.

We now aim to construct a map of $\mathbb{E}_\infty$-rings $K(e^{tC_p}) \to R$ for some R of which we know it has height at least n. This will be done via the Dennis trace map, THH and our previous observations on reversing blueshift.
Recall: Let A be an $\mathbb{E}_\infty$-ring and B an $\mathbb{E}_\infty$-ring with an action of S^1. Then,

$$\mathrm{Map}_{\mathrm{Fun}(BS^1, CAlg)}(THH(A), B) \simeq \mathrm{Map}_{CAlg}(A, Fgt(B))$$

where $Fgt : \mathrm{Fun}(BS^1, CAlg) \to CAlg$ is induced by $f : \star \to BS^1$ and forgets the S^1-action.

Indeed, the left adjoint to Fgt is left Kan extended from $\star \subset BS^1$. Thus, it is given by [5] $\mathrm{colim}_{x \in BS^1}((\star \times_{BS^1} BS^1/x) \simeq S^1 \to \star \xrightarrow{A} CAlg) := \mathrm{colim}_{S^1}(A) \simeq THH(A)$ where the last equivalence is specific to $\mathbb{E}_\infty$-rings.

We apply this recollection to $A = B = e^{tC_p}$ and endow B with the residual $S^1/C_p \cong S^1$ action. Under the above correspondence, the identity of e^{tC_p} corresponds to an S^1-equivariant ring map $THH(e^{tC_p}) \to e^{tC_p}$. Precomposing with the (circle invariant) Dennis trace map and taking fixed points, this yields a ring map

$$K(e^{tC_p}) \to THH(e^{tC_p})^{hS^1/C_p} \to (e^{tC_p})^{hS^1/C_p} \simeq e_p^{tS^1}$$

where the last equivalence is lemma 2.5. By proposition 2.6, the last ring has non-trivial $T(n)$-localisation and thus so does $K(e^{tC_p})$ since there cannot be ring maps from the zero ring to a non-trivial one.

□

3 Redshift for Lubin-Tate theories ([13, Sec. 3])

We now turn to the proof of theorem 1.4. Again, we aim to construct an $\mathbb{E}_\infty$-ring map $E_n \to A$ for some well-chosen $\mathbb{E}_\infty$-ring A of which we know that $L_{T(n+1)}K(A) \not\simeq 0$. Luckily, it is not too hard to construct ring maps out of a Lubin-Tate theory.

Theorem 3.1 (Goerss-Hopkins-Miller, Lurie, [13, Thm. 4.2])**.** *Let $E_n = E_{k,\mathbb{G}_0}$ be a Lubin-Tate theory of height n, R a 2-periodic complex oriented $K(n)$-local $\mathbb{E}_\infty$-ring spectrum. Let $\mathbb{G}_R := \mathrm{Spf}(R^0(\mathbb{C}P^\infty))$ denote the canonical Quillen formal group law over $\pi_0(R)$ and let I_n be the ideal in $\pi_0(R)$ corresponding to the n^{th} Landweber ideal generated by (a choice of) $p, v_1, \dots, v_n$. Then, there is a homotopy equivalence*

$$\mathrm{Map}_{CAlg}(E_n, R) \simeq \{(f,\alpha) \mid f : k \to \pi_0(R)/I_n \textit{ ring hm, } \alpha : f^*\mathbb{G}_0 \cong \mathbb{G}_{R\pi_0(R)/I_n} \textit{ iso over } \pi_0(R)/I_n\}.$$

Thus, we would like to choose A 2-periodic complex oriented, $K(n)$-local to obtain the desired ring map by simply specifying a ring homomorphism $k \to \pi_0(A)/I_n$ and an appropriate isomorphism of formal groups. In view of the previous section, a natural candidate for A would be $L_{K(n)}E_{n+1}^{tC_p} := R'$. This ring does have an associated formal group of height n at certain residue fields. However, these fields are often finite extensions of Laurent series ring over k. The associated formal groups are complicated and not usually isomorphic to another height n formal group over k. It would be nice to modify R' slightly, so that $\pi_0(R')/I_n$ is separably closed and thus all formal groups of a given height are isomorphic over it[6]. This is achieved by the following construction:

Construction 3.2. Let R be an $\mathbb{E}_\infty$-ring spectrum and $\mathfrak{p} \in \mathrm{Spec}(\pi_0(R))$. Then one can produce a ring $\pi_0(R)_{\mathfrak{p}}^{sh}$, the *strict henselization* of $\pi_0(R)$ at $\mathfrak{p}$, such that:

1. $\pi_0(R)_{\mathfrak{p}}^{sh}$ is strictly henselian; in particular, it is a local ring with separably closed residue field.

2. $\pi_0(R)_{\mathfrak{p}}^{sh}$ is a filtered colimit of étale $\pi_0(R)$-algebra maps.

3. $\mathfrak{p}\pi_0(R)_{\mathfrak{p}}^{sh}$ is its maximal ideal.

This construction can be lifted (essentially uniquely) to produce an $\mathbb{E}_\infty$-ring $R_{\mathfrak{p}}^{sh}$ and a ring map $R \to R_{\mathfrak{p}}^{sh}$ with the above properties on π_0.

[5] see [9, Tag 02Y1]
[6] see for example [10, Cor. 15.4] for a discussion of this.

This construction will be useful due to the following facts.

Fact 3.3. 1. Since the residue field is separably closed, all formal groups of height n over a strictly henselian ring are isomorphic.

2. $R_\mathfrak{p}^{sh}$ is a filtered colimit of étale neighborhoods of $\mathfrak{p}$, i.e. $R_\mathfrak{p}^{sh} = \mathrm{colim}_{i\in I} R_\mathfrak{p}^{(i)}$ where each $R_\mathfrak{p}^{(i)}$ is an étale neighborhood of $\mathfrak{p}$.

3. Let Y denote an étale cover of R. The functor $Y \mapsto L_{T(n)}K(Perf(Y))$ is an étale sheaf on R.

Proof of Theorem 1.4. • As already introduced, consider $E_{n+1} = E_{k,\mathbb{G}_0'}$ a Lubin-Tate theory over k of height $n+1$ and set $R := E_{n+1}^{tC_p}$. By theorem 2.8, we know that $L_{T(n+1)}K(R) \not\simeq 0$.

- Claim 1: There exists $\mathfrak{p} \in \mathrm{Spec}(\pi_0(R))$ such that $L_{T(n+1)}K(R_\mathfrak{p}^{sh}) \not\simeq 0$.

  In what follows, we consider $R_\mathfrak{p}^{sh}$ for this specific $\mathfrak{p}$. We defer the proof of this claim to the end of the section.

- Since $L_{T(n+1)}R \simeq 0$ by blueshift (thm. 2.1) [7], $L_{T(n)\oplus T(n+1)}R_\mathfrak{p}^{sh} \simeq L_{T(n)}R_\mathfrak{p}^{sh}$ [8]. Combining this with the tautology that $R_\mathfrak{p}^{sh} \to L_{T(n)\oplus T(n+1)}R_\mathfrak{p}^{sh}$ is a $T(n)\oplus T(n+1)$-equivalence and theorem 2.9 yields $L_{T(n+1)}K(L_{T(n)}R_\mathfrak{p}^{sh}) \simeq L_{T(n+1)}K(R_\mathfrak{p}^{sh}) \not\simeq 0$, where the last non-equivalence is claim 1.

- By a result of Hovey ([6, Cor. 1.10]), $L_{T(n+1)}K(L_{T(n)}R_\mathfrak{p}^{sh}) \simeq L_{T(n+1)}K(L_{K(n)}R_\mathfrak{p}^{sh})$. The ring $L_{K(n)}R_\mathfrak{p}^{sh}$ is the one that will play the role of A from the first paragraph of this section.

- It remains to construct a map of $\mathbb{E}_\infty$-rings $E_n = E_{k,\mathbb{G}_0} \to L_{K(n)}R_\mathfrak{p}^{sh}$. Then, we conclude by the previous point and the, by now familiar, fact that there cannot be a ring map from the zero ring to a non-trivial one.

- Set $\Gamma := \mathrm{Spf}((R_\mathfrak{p}^{sh})^0(\mathbb{C}P^\infty))$. By theorem 3.1, to get a ring map $E_n \to L_{K(n)}R_\mathfrak{p}^{sh}$, it suffices to construct a ring homomorphism $f : k \to \pi_0(R_\mathfrak{p}^{sh})/I_n$ and an isomorphism of formal groups $f^*(\mathbb{G}_0) \cong \Gamma_{\pi_0(R_\mathfrak{p}^{sh})/I_n}$.

- To construct f, recall that $\pi_*(R) = W(k)[v_1,\dots,v_n][u^\pm]((x))/([p](x))$ with u and x in degree 2, the rest in degree 0. In particular, the inclusion of k into its Witt vectors yields a ring map $k \to \pi_0(R)/I_n$. Composing this with $\pi_0(R)/I_n \to \pi_0(R_\mathfrak{p}^{sh})/I_n$ gives f.

- Claim 2: Γ has height n and $\pi_0(R_\mathfrak{p}^{sh})/I_n$ is strictly henselian.

  The first part of this claim is essentially lemma 4.6 in [13] (and was skipped for time reasons). Roughly, the idea is to compute the height at the residue field and observe that it cannot be greater than n because R (and thus $R_\mathfrak{p}^{sh}$) is of height n, nor smaller than n because that would contradict $K(R_\mathfrak{p}^{sh})$ having height $n+1$ (via theorem 2.9). The second part follows from the fact that I_n is contained in $\mathfrak{p}$.

- We conclude by fact 3.3.1, that the required isomorphism exists and thus conclude the proof of theorem 1.4.

□

[7] As there is a ring map from R to its strict henselization, this implies $L_{T(n+1)}R_\mathfrak{p}^{sh} = 0$.

[8] It is not always true that for ring spectra A, B, X such that $X\otimes B \simeq 0$, it holds that $L_{A\vee B}X \simeq L_A X$. This holds if and only if $L_A X$ is also B-acyclic. This is always guaranteed if L_A is smashing or if the localisation $X \to L_A X$ is a ring map (which is the situation here). However, set $A = \mathbb{S}/p$, $B = \mathbb{Q}$ and $X = \mathbb{Q}/\mathbb{Z}$. Then, $B\otimes X = 0$, but $L_A X = \Sigma\mathbb{Z}_p^\wedge$ which is not $\mathbb{Q}$-acyclic.

We conclude this section by giving the proof of claim 1, because it illustrates nicely how to make use of étale descent.

Proof of Claim 1. We proceed by contradiction, aiming to contradict the fact that $L_{T(n+1)}R \not\simeq 0$. Suppose that $L_{T(n+1)}K(R^{sh}_{\mathfrak{p}}) \simeq 0$ for all $\mathfrak{p} \in \mathrm{Spec}(\pi_0(R))$. Recall from fact 3.3.2 that $R^{sh}_{\mathfrak{p}}$ is a filtered colimit of étale neighborhoods of $\mathfrak{p}$, $R^{sh}_{\mathfrak{p}} = \mathrm{colim}_{i\in I} R^{(i)}_{\mathfrak{p}}$. The functor $K(-)$ commutes with filtered colimits, thus

$$L_{T(n+1)}K(R^{sh}_{\mathfrak{p}}) \simeq 0 \iff L_{T(n+1)} \mathrm{colim}_{i\in I} K(R^{(i)}_{\mathfrak{p}}) \simeq 0 \iff \mathrm{colim}_{i\in I} T(n+1) \otimes R^{(i)}_{\mathfrak{p}} \simeq 0.$$

As in lemma 1.8, $T(n+1)$ can be chosen to have a ring structure and the above colimit is a filtered colimit of rings. This vanishes if and only if it vanishes at a finite stage [9]. Thus, for each $\mathfrak{p} \in \mathrm{Spec}(\pi_0(R))$ there exists some $i_p \in I$ such that $L_{T(n+1)}K(R^{(i_p)}_{\mathfrak{p}}) \simeq 0$. Now, we invoke fact 3.3.3 (K-theory is an étale sheaf) and the fact that these $\{R^{(i_p)}_{\mathfrak{p}}\}_{\mathfrak{p}\in\mathrm{Spec}(\pi_0(R))}$ are an étale cover of R to conclude that $L_{T(n+1)}R \simeq 0$. A contradiction. □

4 Redshift for iterated K-theory of fields ([13, Sec. 4])

Finally, we sketch the proof of theorem 1.5. For convenience, let us recall the statement here. Throughout, for any functor F that can be iterated, $F^{(n)}$ will denote the n-fold iterate of F.

Theorem 4.1. *Let $n \geq 0$, for any field k of characteristic different from p, the spectrum $L_{T(n)}K^{(n)}(k)$ is nonzero.*

Proof. We begin by reducing the statement to the case $k = \mathbb{Q}(\xi_p)$ with ξ_p a p^{th}-root of unity.

- First, reduce to the case when k is algebraically closed. (There is a ring map from k to its algebraic closure, then apply the usual trick).
- In that case, Suslin (1983/84, [11], [12]) showed that $L_{T(1)}K(k) \simeq KU_p$.
- By a result of Bhatt-Clausen-Mathew (2020, [2]): $L_{T(1)}K(\mathbb{Q}(\xi_{p^\infty})) \simeq L_{T(1)}(KU_p \otimes K(\mathbb{Q}))$. By tensoring with the unit and applying $L_{T(1)}K(-)$, the latter admits a ring map from KU_p. By a series of applications of theorem 2.9 and 1.3, this allows one to reduce to the case $k = \mathbb{Q}(\xi_{p^\infty}) := \mathrm{colim}_j \mathbb{Q}(\xi_{p^j})$ [10].
- As in the proof of claim 1, we use that K-theory commutes with filtered colimits to get $T(n) \otimes K^{(n)}(\mathbb{Q}(\xi_{p^\infty})) \simeq colim_j T(n) \otimes K^{(n)}(\mathbb{Q}(\xi_{p^j}))$. As a filtered colimit of rings, this vanishes if and only if it vanishes at a finite stage. Thus, we are left to show that $T(n) \otimes K^{(n)}(\mathbb{Q}(\xi_{p^j})) \not\simeq 0$ for all nonnegative integers j.
- This step uses the magic of Galois extensions of $\mathbb{E}_\infty$-rings (in the sense of Rognes) to reduce to the case $k = \mathbb{Q}(\xi_p)$. Observe that $\mathbb{Q}(\xi_p) \subset \mathbb{Q}(\xi_{p^j})$ is a (classical) Galois extension with Galois group $G := C_{p^{j-1}}$.

[9] This colimit vanishes if and only if the unit of the colimit is trivial. The unit is an element in $\mathrm{Map}(\mathbb{S}, \mathrm{colim}_i R^i) \simeq \mathrm{colim}_i \mathrm{Map}(\mathbb{S}, R^i)$ using compactness of the sphere spectrum. Under this equivalence, the unit is the colimit of the units. This vanishes if and only if there exists i such that the unit of R^i is trivial. But this implies that the ring itself is trivial, thus the colimit vanishes at a finite stage.

[10] For example, to show that $L_{T(2)}K^{(2)}(k) \neq 0$, we observe that by theorem 2.9 the map $K^{(2)}(k) \to K(L_{T(1)\vee T(2)}K(k))$ is a $T(2)$-equivalence. Via the ring map $L_{T(2)}K(L_{T(1)\vee T(2)}K(k)) \to L_{T(2)}K(L_{T(1)}K(k))$ and Suslin's computation, we reduce to showing that $L_{T(2)}K(KU_p) \neq 0$. By Bhatt-Clausen-Matthew's result, this is the case if $L_{T(2)}K(L_{T(1)}K(\mathbb{Q}(\xi_{p^\infty}))) \neq 0$. This is true if $L_{T(2)}K^{(2)}(\mathbb{Q}(\xi_{p^\infty})) \neq 0$. Indeed, by theorem 2.9, $L_{T(2)}K^{(2)}(\mathbb{Q}(\xi_{p^\infty})) \simeq L_{T(2)}K(L_{T(1)\vee T(2)}K(\mathbb{Q}(\xi_{p^\infty})))$. As $\mathbb{Q}(\xi_{p^\infty})$ is rational, by theorem 1.3, $K(\mathbb{Q}(\xi_{p^\infty}))$ is $T(2)$-acyclic and by footnote 7, $L_{T(2)}K(L_{T(1)\vee T(2)}K(\mathbb{Q}(\xi_{p^\infty}))) \simeq L_{T(2)}K(L_{T(1)}K(\mathbb{Q}(\xi_{p^\infty})))$.

Claim: This remains a G-Galois extension after applying $L_{T(n)}K^{(n)}(-)$. In particular, $L_{T(n)}K^{(n)}(\mathbb{Q}(\xi_p)) \simeq L_{T(n)}K^{(n)}(\mathbb{Q}(\xi_{p^j}))^{hC_{p^{j-1}}}$. We will not discuss the proof here, this is theorem 5.12 in [13] and essentially a combination of results from [7], [4] and [1].

The case $k = \mathbb{Q}(\xi_p)$ is shown by refining the case $k = \mathbb{Q}$ (which we will do in the next subsection). More precisely, in the proof of the rational case, we will construct some ring map $\mathbb{Q} \xrightarrow{\psi} R \otimes \mathbb{Q}$ such that $K^{(n)}(R\otimes\mathbb{Q})$ is not $T(n)$-acyclic. One can show that there is a ring map $\Phi^{C_p^{\times n}} e_n \to R$, where e_n is the connective cover of some Lubin-Tate theory of height n with algebraically closed residue field. By [13, Prop. 5.8], this implies that $\pi_0(R \otimes \mathbb{Q})$ contains a primitive p^{th}-root of unity. Using that $\mathbb{Q} \to \mathbb{Q}(\xi_p)$ is étale, one can extend ψ to a ring map $\mathbb{Q}(\xi_p) \to R \otimes \mathbb{Q}$ to conclude. □

4.1 The case $k = \mathbb{Q}$

To show theorem 1.5 in the case $k = \mathbb{Q}$, we proceed in two familiar steps. We first construct a map of $\mathbb{E}_\infty$-rings $L_{T(n)}K^{(n)}(\mathbb{Q}) \to L_{T(n)}K^{(n)}(R)$ for some well chosen ring R and then show that $K^{(n)}(R)$ is not $T(n)$-acyclic. In view of the previous sections, it makes sense to choose R to be some iterated variant of E_n and its Tate construction. We set $t_+E_n := (\tau_{\geq 0}E_n)^{tC_p}$ and set $R = t_+^{(n)}E_n$.

Lemma 4.2. *There exists an $\mathbb{E}_\infty$-ring map $L_{T(n)}K^{(n)}(\mathbb{Q}) \to L_{T(n)}K^{(n)}(R)$.*

Proof. Theorem 2.9 implies that $L_{T(n)}K^{(n)}(R) \simeq L_{T(n)}K(L_{T(n)\vee T(n-1)}K^{(n-1)}(R))$. Iterating this, one finds that

$$L_{T(n)}K^{(n)}(R) \simeq L_{T(n)}K^{(n)}(L_{T(0)\vee\cdots\vee T(n-1)\vee T(n)}R).$$

Claim: R is $T(1) \vee \cdots \vee T(n-1) \vee T(n)$-acyclic, so

$$L_n^{p,f}R := L_{T(0)\vee\cdots\vee T(n-1)\vee T(n)}R \simeq L_{T(0)}R.$$

This can be shown computationally and Yuan also gives a more hands-off proof using genuine equivariant homotopy theory ([13, Lem. 5.3]).
By this claim, $L_n^{p,f}R$ is rational (we work p-completed), so it is a $\mathbb{Q}$-module. Thus, the unit map $\mathbb{Q} \to L_n^{p,f}R$ will give the desired ring map- □

Lemma 4.3. $L_{T(n)}K^{(n)}(R) \not\simeq 0$.

Proof. The proof is similar to that of theorem 2.8. Analogous adjunctions to those in the proof of theorem 2.8 combined with the $(S^1)^{\times n}$-equivariant Dennis trace map give a ring map

$$K^{(n)}(R) \to THH^{(n)}(R)^{h(S^1)^{\times n}} \to R^{h(S^1/C_p)^{\times n}}.$$

Thus, all that is left to show is that $R^{h(S^1/C_p)^{\times n}}$ is not $T(n)$-acyclic.

To do so, we show inductively on $k \geq 0$ that $(t_+^{(k)}(E_n))^{h(S^1/C_p)^{\times k}}$ is not $T(n)$-acyclic. The idea is, as before, to use the Tate-Orbit lemma to make $(-)^{tS^1}$ appear and exploit that this does not lower the height.
The case $k = 0$ is clear. Thus, let $k \geq 1$ and suppose the statement was shown for $k-1$. Recall that

$$t_+^{(k)}(E_n) = t_+(t_+^{(k-1)}(E_n)) = (\tau_{\geq 0}t_+^{(k-1)}(E_n))^{tC_p^{(k)}}\text{[11]}.$$

Using the Tate-Orbit lemma (lemma 2.5) in the last equality:

$$(t_+^{(k)}(E_n))^{hS^1/C_p^{(k)}} = ((\tau_{\geq 0}t_+^{(k-1)}(E_n))^{tC_p^{(k)}})^{hS^1/C_p^{(k)}} \simeq_p (\tau_{\geq 0}t_+^{(k-1)}(E_n))^{tS^1}.$$

[11] By this notation, we mean the action of the k^{th} C_p group in the product.

Now $\tau_{\geq 0} t_+^{(k-1)}(E_n)$ is complex oriented, thus, as in the proof of proposition 2.6, it is a retract of its S^1-Tate construction. Thus, applying fixed points to the previous equivalences,

$$(t_+^{(k)}(E_n))^{h(S^1/C_p)^{\times k}} \simeq ((t_+^{(k)}(E_n))^{hS^1/C_p^{(k)}})^{h(S^1/C_p)^{\times k-1}} \simeq_p ((\tau_{\geq 0} t_+^{(k-1)}(E_n))^{tS^1})^{h(S^1/C_p)^{\times k-1}}.$$

The last term has $\tau_{\geq 0} t_+^{(k-1)}(E_n)^{h(S^1/C_p)^{\times k-1}}$ as a retract and this is not $T(n)$-acyclic by inductive assumption. □

References

[1] T. Barthel, S. Carmeli, T. M. Schlank, and L. Yanovski. The chromatic fourier transform, 2022. URL `https://arxiv.org/abs/2210.12822`.

[2] B. Bhatt, D. Clausen, and A. Mathew. Remarks on $k(1)$-local k-theory, 2020. URL `https://arxiv.org/abs/2001.11622`.

[3] R. Burklund, T. M. Schlank, and A. Yuan. The chromatic nullstellensatz, 2022. URL `https://arxiv.org/abs/2207.09929`.

[4] D. Clausen, A. Mathew, N. Naumann, and J. Noel. Descent and vanishing in chromatic algebraic k-theory via group actions, 2022. URL `https://arxiv.org/abs/2011.08233`.

[5] J. Hahn. On the bousfield classes of h_∞-ring spectra, 2022. URL `https://arxiv.org/abs/1612.04386`.

[6] M. Hovey. Bousfield localization functors and Hopkins' chromatic splitting conjecture. In *The Čech centennial. A conference on homotopy theory dedicated to Eduard Čech on the occasion of his 100th birthday, June 22-26, 1993, Northeastern University, Boston, MA, USA*, pages 225–250. Providence, RI: American Mathematical Society, 1995. ISBN 0-8218-0296-8.

[7] M. Land, A. Mathew, L. Meier, and G. Tamme. Purity in chromatically localized algebraic k-theory, 2023. URL `https://arxiv.org/abs/2001.10425`.

[8] J. Lurie. Chromatic homotopy theory, 2010. URL `https://people.math.harvard.edu/~lurie/252x.html`.

[9] J. Lurie. Kerodon. `https://kerodon.net`, 2025.

[10] P. Pstragowski. Finite height chromatic homotopy theory, 2021. URL `https://www.kurims.kyoto-u.ac.jp/~piotr/252y_notes.pdfl`.

[11] A. Suslin. On the K-theory of algebraically closed fields. *Invent. Math.*, 73:241–245, 1983. ISSN 0020-9910. doi: 10.1007/BF01394024. URL `https://eudml.org/doc/143044`.

[12] A. A. Suslin. On the K-theory of local fields. *J. Pure Appl. Algebra*, 34:301–318, 1984. ISSN 0022-4049. doi: 10.1016/0022-4049(84)90043-4.

[13] A. Yuan. Examples of chromatic redshift in algebraic K-theory. Preprint, arXiv:2111.10837 [math.KT] (2021), 2021. URL `https://arxiv.org/abs/2111.10837`.