The Hopf invariant one problem or when is there a division algebra structure on $\mathbb{R}^n$?

Notes to a talk I gave for the graduate seminar on topology at the university of Bonn organised by Prof. Dr. Stefan Schwede and Dr. Jack Davies in the winter semester 2023/24

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1 Introduction

Before diving into the maths, let us jump back to 19th century Ireland where a certain Sir William Hamilton was trying to "multiply triplets". By triplets, he meant what we nowadays refer to as three-dimensional real vectors (i.e. an element of the vector space $\mathbb{R}^3$) and by multiplication, a distributive law without zero divisors. Every morning at breakfast his sons asked him "Well, Papa, can you multiply triplets?" and every morning he was forced to reply "No, I can only add and subtract them" (see [8] for more details). Eventually, he gave up on triplets and focused on the situation in four dimensions. There, he was more successful and discovered the quaternions. In these notes, we will learn why Hamilton could never have succeeded in multiplying triplets. Indeed, we will explain the proof of:

Theorem 1.1. *Unless $n = 1, 2, 4, 8$, there exists no real division algebra structure on $\mathbb{R}^n$.*

To make sure we are all on the same page, let us recall the definition of real division algebra.

Definition 1.2. A real division algebra structure on $\mathbb{R}^n$ is a bilinear (with respect to addition and scalar multiplication coming from the vector space structure), continuous map

$$\mu : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$$

such that for all $x \in \mathbb{R}^n - \{0\}$ the maps $\mu(x, -)$ and $\mu(-, x)$ are injective (and hence, for dimension reasons, invertible). In other words, the existence of zero divisors is not allowed.

Remark 1.3.

1. Even though it might be unclear from the definition, the terminology division algebra makes sense as we can modify μ so that there is a unit and all non-zero elements are invertible with respect to that unit. Indeed, choose any unit vector $e \in \mathbb{R}^n$ and construct a linear isomorphism $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ taking $\mu(e, e)$ to e. This can be done for example by completing both $\{e\}$ and $\{\mu(e, e)\}$ to bases of $\mathbb{R}^n$ and defining φ by mapping $\mu(e, e)$ to e and pairing the other basis elements in any way. By composing with φ, we may, without loss of generality, assume that $\mu(e, e) = e$. Now, for simplicity of notation write $\alpha = \mu(e, -)$ and $\beta = \mu(-, e)$. Recall that both these maps are invertible. Finally, the map
$$\tilde{\mu}(x, y) := \mu(\beta^{-1}(x), \alpha^{-1}(y))$$
satisfies $\tilde{\mu}(x, e) = \mu(\beta^{-1}(x), e) = \beta \circ \beta^{-1}(x) = x$ and similarly $\tilde{\mu}(e, y) = y$. Hence, e is the unit for $\tilde{\mu}$. Moreover, the maps $\tilde{\mu}(x, -)$ and $\tilde{\mu}(-, y)$ are injective as composition of injective maps. Hence, for dimensional reasons, they are surjective. In particular, there exist some elements $a, b \in \mathbb{R}^n$ such that $\tilde{\mu}(x, a) = e = \tilde{\mu}(b, x)$ i.e. left and right inverses exist. The observation that we can without loss of generality assume μ to be unital will be important in what follows when we connect this result to topology.

2. The requirement that there are no zero divisors is what makes it hard to find such a structure. Indeed, this is why the naive approach of coordinate wise multiplication for example does not induce a division algebra structure.

3. Observe that we do not require commutativity or even associativity of the multiplication. Moreover, in the case $n = 4$ an example of such a division algebra structure is given by the quaternions ($\mathbb{H}$) which are not commutative and for $n = 8$ there are the octonions ($\mathbb{O}$) which are not even associative. Clearly, for $n = 1$ the field structure of $\mathbb{R}$ is a division algebra structure and for $n = 2$ the complex numbers work. In particular, this theorem cannot be further refined.

This is all nice and well, but one question lingers around. These notes are based on a talk for a seminar in topology, so where does this come in?
The link to topology is easy to come across : any division algebra structure μ on $\mathbb{R}^n$ induces an H-space structure (continuous multiplication with two-sided identity element potentially up to homotopy) on the unit sphere S^{n-1} by defining

$$\mu' : S^{n-1} \times S^{n-1} \to S^{n-1}, \ \mu'(x, y) = \frac{\mu(x, y)}{|\mu(x, y)|}.$$

The non-existence of zero divisors for μ ensures this is well-defined and unitality of μ (see remark 1.3) ensures μ' is really an H-space multiplication rather than just some continuous map. Thus, if we can show that S^{n-1} admits an H-space structure only for $n = 1, 2, 4, 8$, we have also shown theorem 1.1. We will from now on be concerned with the question: for which n does S^{n-1} admit an H-space structure?
This was (to my knowledge) first completely answered by Adams in [1] using advanced methods from cohomology theory. Adams himself states that the results he uses are not very deep but buried in technicalities. In these notes, we will be focusing on an alternate, less technical proof published by Adams and Atiyah in [2] using K-theory. In one sentence, we will use relations

appearing in this extraordinary cohomology theory to impose restrictions on the possible values of n.

The plan for the coming pages is as follows:
First, there will be a brief section about vector bundles, which play a similar role in K-theory as simplicial cochains do in singular cohomology. In fact, K-theory will be all about putting a ring structure on certain sets of vector bundles. Then, we will define K-theory and state a few useful facts without proofs. Having done this, we are in possession of all the necessary tools to reformulate the question of finding H-space structures in a more convenient way, using the notion of Hopf invariant. At last, we introduce the Adams operations (the relations we will need to pin down the values of n), prove the reformulated version of theorem 1.1 and sketch the construction of the Adams operations.
In what follows, and already in the previous, we mainly follow [5] in particular section 2.3, but some inspiration also comes from [3].

2 Vector Bundles

In this section, we discuss the main objects we need for constructing K-theory: vector bundles.

2.1 First Definitions ([3, Section 1.1] and [5, Section 1.1])

Definition 2.1. A (complex) vector bundle over a space X is a continuous map $p : E \to X$ together with a finite dimensional (complex) vector space structure on each fibre $p^{-1}(x)$. Moreover, we require that for every $x \in X$ there exists an open neighbourhood U_x, some positive integer n_x and a homeomorphism $h_x : p^{-1}(U_x) \to U_x \times \mathbb{C}^{n_x}$ which is a linear isomorphism on each fibre. These are called local trivialisations.

Remark 2.2.
- We often simply write E to denote a vector bundle $p : E \to X$. We call X the base space and E the total space.
- The local triviality condition forces n_x, i.e. the dimension of the fibre, to be constant on every connected component of X.
- A vector bundle with fiber dimension 1 is called line bundle and often denoted L.

Definition 2.3. Two vector bundles $p_1 : E_1 \to X$ and $p_2 : E_2 \to X$ are said isomorphic if there exists a homeomorphism $\eta : E_1 \to E_2$ that restricts to a linear isomorphism on each fiber and such that $p_2 \circ \eta = p_1$. This is denoted by $E_1 \cong E_2$.

From now on, for simplicity, all base spaces X will be assumed to be compact CW-complexes. For our purposes, this restriction causes no problems as eventually we will be interested in spheres. The interested reader can find certain statements in more generality in [5]. Moreover, unless explicitly stated otherwise, all vector spaces will be assumed to be over $\mathbb{C}$.
Let us look at some examples of vector bundles that will be needed over and over again.

Example 2.4.
1. *Trivial n-bundle*: (denoted ϵ^n or sometimes even n)
$$p_X : \mathbb{C}^n \times X \to X$$
where p_X denotes projection onto X. It is obvious that this is a vector bundle.
2. *Canonical or tautological line bundle:*
View complex projective n-space as the equivalence classes of lines in $\mathbb{C}^{n+1}$, denote its elements by l or in homogeneous coordinates by $[a_0 : \cdots : a_n]$. Consider the subset
$$E := \{(l, v) \mid v \in l\} \subset \mathbb{C}P^n \times \mathbb{C}^{n+1}$$

i.e. E is the set of all points in complex projective space which we identify with lines and v are the vectors that are on these lines. Define the canonical line bundle

$$p : E \to \mathbb{C}P^n,\ (l, v) \mapsto l.$$

To check that this is indeed a line bundle, we have to specify the local trivialisations. As is common when working with projective space, choose as open cover the standard affine opens $D_k = \{[a_0 : \cdots : a_n] \mid a_k \neq 0\}$. Without loss of generality, we can represent any element of D_k in homogeneous coordinates with $a_k = 1$ i.e. on D_k we set $[a_0 : \cdots : a_n] = [\frac{a_0}{a_k} : \cdots : 1 : \cdots : \frac{a_n}{a_k}] := a$. Then, on D_k, $a \mapsto \frac{a_i}{a_k}$ is well-defined and we can define the homeomorphisms by

$$h_k : p^{-1}(D_k) \to D_k \times \mathbb{C},\ (l, v) \mapsto (l, v_k)$$

where v_k is the k^{th} coordinate of v (enumerating the coordinates starting at 0). An explicit inverse is given by

$$f : D_k \times \mathbb{C} \to p^{-1}(D_k),\ f(a, t) = (a, (t\frac{a_0}{a_k}, \ldots, t, \ldots, t\frac{a_n}{a_k})).$$

The key observation to see that this is really an inverse is to observe that since $v \in l$, v must be equal to $(ta_0, \ldots, t, \ldots, ta_n)$ where the a_i are the homogeneous coordinates of l such that $a_k = 1$ and t is some complex number.

In what follows, we will often use H to denote the canonical line bundle on $\mathbb{C}P^1 \cong S^2$.

3. In order to provide some more geometric intuition, let us for the purpose of this example consider real line bundles on the circle. It is easy to see that the trivial real line bundle over the circle coincides with the tangent bundle. A non-trivial example of a real line bundle over S^1 is given by the Möbius strip. The reader is invited to check this and to also check the fact that the Möbius strip corresponds to the canonical line bundle when identifying S^1 with $\mathbb{R}P^1$. On the other hand, there are no non-trivial complex vector bundles over S^1. For a proof (that requires some machinery that hasn't been introduced yet) see for example [7]. This fact will play a key role in what follows (see for example 3.11). The reader is encouraged to look for other places where working with complex numbers is key.

2.2 Operations on vector bundles ([3, Section 1.2] and [5, p. 9-15])

One reason why vector bundles are nice is that many of the operations we can perform on vector spaces carry over to vector bundles. This will later allow us to give a ring structure to some equivalence class of vector bundles over X (some base space) and introduce a notion of functoriality. All of this will be made precise in the following section on K-theory. For now, let us introduce the main operations. Fix a (compact, CW-complex) base space X and consider two vector bundles $p_1 : E_1 \to X$, $p_2 : E_2 \to X$ over X. We define:

Definition 2.5. The direct sum $E_1 \oplus E_2$ of the vector bundles E_1 and E_2 is given by

$$p : \{(v, v') \mid p_1(v) = p_2(v')\} \subset E_1 \times E_2 \to X,\ (v, v') \mapsto p_1(v).$$

It is left to the reader to check that this is indeed a vector bundle. A hint for an elegant proof is to realise the $E_1 \oplus E_2$ is the restriction of the product bundle

$$E_1 \times E_2 \to X \times X,\ (v, v') \mapsto (p_1(v), p_2(v'))$$

to the diagonal of X and that it is straightforward to check that the product bundle and restriction of vector bundles are vector bundles. The motivation for defining the direct sum in

this way is the fact that the fiber of $E_1 \oplus E_2$ is the direct sum of the fibres of E_1 and E_2 i.e. $p^{-1}(x) = p_1^{-1}(x) \oplus p_2^{-1}(x)$.
Next we define:

Definition 2.6. The tensor product $E_1 \otimes E_2$ of the vector bundles E_1 and E_2 is defined as a set by

$$E_1 \otimes E_2 = \sqcup_{x \in X} p_1^{-1}(x) \otimes_{\mathbb{C}} p_2^{-1}(x)$$

with the obvious projection map to X. We force this to be a vector bundle by defining the topology on $E_1 \otimes E_2$ as follows: for any open set U such that both E_1 and E_2 are trivial on U we choose, for $i = 1, 2$, trivialisations $h_U^i : p_i^{-1}(U) \to U \times \mathbb{C}^{n_i}$ and require the tensor product

$$h_U^1 \otimes h_U^2 : p_1^{-1}(U_1) \otimes p_2^{-1}(U) \to U \times (\mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2})$$

to be a homeomorphism. We leave to the reader to check that this is well-defined.

In a similar way, we can define the exterior power of a vector bundle. Recall that for a vector space V the i^{th} exterior power $\lambda^i(V)$ is given by

$$\lambda^i(V) = V \otimes \cdots \otimes V / \langle v_1 \otimes \cdots \otimes v_i - sgn(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(i)} \rangle$$

for all permutations σ.

Definition 2.7. The i^{th} exterior power $\lambda^i(E_1)$ of E_1 is defined as a set as $\lambda^i(E_1) = \sqcup_{x \in X} \lambda^i(p_1^{-1}(x))$ with the obvious projection map onto X. We make this into a vector bundle by defining a topology in an analogous way to definition 2.6 (see [3, p. 6-8]).

The final construction encodes the functoriality of vector bundles with respect to maps on the base space.

Definition 2.8. Consider a continuous map $f : Y \to X$. We define the pullback bundle $\pi : f^*(E_1) \to Y$ where $f^*(E_1) = \{(e, y) \mid p_1(e) = f(y)\}$ is the usual pull-back and π the projection onto Y. The vector space structure on the fibres is inherited from p_1 by observing that $\pi^{-1}(y) \cong p_1^{-1}(f(y))$. One can check that the local trivialisations are given by the preimages under f of the local trivialisations of $p_1 : E_1 \to X$.

The pullback construction is functorial (or, to be pedantic, pseudo-functorial) as it only holds up to isomorphism) i.e. for two maps $f : Y \to X$, $g : Z \to Y$ $(f \circ g)^* \cong g^* f^*$. It commutes with direct sums, tensor products and exterior powers. Moreover, we have the usual properties and distributivity relations between direct sums and tensor products, similarly for exterior powers. Note also that $\oplus$ and $\otimes$ are commutative up to isomorphism, see [5, section 1.1] for more details).

3 K-theory

We are now in the position to define K-theory and to state some useful facts we will use later.

3.1 K-theory Basics ([5, Section 2.1])

Definition 3.1 ([5, p. 39-40])**.** For a compact CW-complex X, we define $K(X)$ to be the set of formal differences $E - E'$ where E, E' are vector bundles over X up to being stably isomorphic (denoted $\simeq_s$). By this, we mean that two formal differences $E_1 - E_1'$ and $E_2 - E_2'$ are identified if and only if there exists some positive integer n such that $E_1 \oplus E_2' \oplus \epsilon^n \cong E_2 \oplus E_1' \oplus \epsilon^n$. Recall that ϵ^n denotes the trivial n-bundle.

We will abuse notation and also use $\simeq_s$ to denote vector bundles that are stably isomorphic. More precisely, given two vector bundles E, E' over X, we write $E \simeq_s E'$ if there exists some trivial bundle ϵ^n such that $E \oplus \epsilon^n \cong E' \oplus \epsilon^n$. The equivalence class of the formal difference $E - E'$ is denoted by $[E - E']$. However, if it is clear from the context, we will drop the brackets and simply write $E - E'$ also for the equivalence class for simplicity.

Remark 3.2. The relation $\simeq_s$ is really an equivalence relation. Reflexivity and symmetry are straightforward. To check transitivity of $\simeq_s$, one needs the following cancellation property:

$$E_1 \oplus E_2 \simeq_s E_1 \oplus E_3 \text{ implies } E_2 \simeq_s E_3.$$

For this to hold, compactness of X is crucial as for any vector bundle E over X we need to be able to find some vector bundle E' such that $E \oplus E' \cong \epsilon^n$ for some positive integer n. One can show that this is always possible for compact, Hausdorff base spaces (see [5, Prop. 1.4]), but fails without compactness; there are counterexamples over $\mathbb{R}P^\infty$ (see [5, Example 3.6]).
Knowing this, the cancellation property follows easily by adding a bundle E_1' such that $E_1' \oplus E_1$ is trivial to both sides. Let us now prove transitivity: suppose $E_1 - E_1' \simeq_s E_2 - E_2'$ and $E_2 - E_2' \simeq_s E_3 - E_3'$ i.e. there exist $m, n \in \mathbb{N}$ such that $E_1 \oplus E_2' \oplus \epsilon^n \cong E_1' \oplus E_2 \oplus \epsilon^n$ and $E_2 \oplus E_3' \oplus \epsilon^m \cong E_2' \oplus E_3 \oplus \epsilon^m$. Adding the two equations together we find that

$$E_1 \oplus E_2' \oplus \epsilon^n \oplus E_2 \oplus E_3' \oplus \epsilon^m \cong E_1' \oplus E_2 \oplus \epsilon^n \oplus E_2' \oplus E_3 \oplus \epsilon^m.$$

Observe that $E_2 \oplus E_2'$ appears on both sides and can be cancelled. Using that $\epsilon^n \oplus \epsilon^m \cong \epsilon^{m+n}$, the equation rewrites as $E_1 \oplus E_3' \oplus \epsilon^{m+n} \cong E_1' \oplus E_3 \oplus \epsilon^{m+n}$ or $E_1 - E_1' \simeq_s E_3 - E_3'$ as desired.

The set $K(X)$ is interesting because of the following proposition.

Proposition 3.3 ([5, p. 39-41])**.** *One can define operations $+$ and $\cdot$ on $K(X)$ that make $K(X)$ into a commutative ring with identity. Moreover, the pullback of vector bundles makes $K(-)$ into a functor from the category of compact CW-complexes to rings.*

Proof. (sketch) We only give the definitions of the operations here. The reader is left to check this has the desired properties and is compatible with the equivalence relation. We define:

- +: $[E_1 - E_1'] + [E_2 - E_2'] := [E_1 \oplus E_2 - E_1' \oplus E_2']$.
- 0: The neutral element is given by the equivalence class $[E - E]$ for any vector bundle E over X.
- -: The inverse of $[E_1 - E_1']$ is, not surprisingly, $[E_1' - E_1]$.
- $\cdot$: $[E_1 - E_1'] \cdot [E_2 - E_2'] := [E_1 \otimes E_2 - E_1 \otimes E_2'] + [E_1' \otimes E_2' - E_1' \otimes E_2]$
- 1: The identity is the trivial line bundle ϵ^1 (i.e. the equivalence class $[\epsilon^1 - \epsilon^0]$).

It remains to define $K(f)$ for $f : Y \to X$ some continuous map. We make the natural definition $K(f) := f^* : K(X) \to K(Y),\ [E - E'] \mapsto [f^*(E) - f^*(E')]$. $\square$

There is also a reduced version of K dependent on the choice of some base point $x_0 \in X$, constructed as follows.

Definition 3.4 ([5, p. 39-40])**.** Let $i : \{x_0\} \to X$ be the inclusion of the chosen base point. We define $\tilde{K}(X) := \ker(K(X) \to K(x_0))$.

Remark 3.5. 1. The reduced K-group consists of those equivalence classes of formal differences of vector bundles that are of the same dimension when restricted to x_0 and thus of the same dimension on the entire connected component containing x_0. Indeed,

$$\ker(K(X) \to K(x_0)) = \{E - E' \mid E|_{x_0} \simeq_s E'|_{x_0}\} =$$
$$\{E - E' \mid dim(E) = dim(E') \text{ on the connected component containing } x_0\}.$$

where the first equality follows directly from the definitions. The second one holds because all vector bundles over a point are trivial by definition, which implies that, over a point, being stably isomorphic is equivalent to being isomorphic which is equivalent to being of the same dimension.

2. As kernel of a ring morphism, $\tilde{K}(X)$ is an ideal of $K(X)$. As such, it inherits a multiplicative ring structure, but without identity.

3. One can explicitly describe $K(x_0)$. Since the only vector bundles over a point are trivial bundles, $K(x_0) = \{\epsilon^n - \epsilon^m \mid m, n \in \mathbb{N}\}/\simeq_s$. Moreover, $K(x_0) \cong \mathbb{Z}$ via $\epsilon^n - \epsilon^m \mapsto n - m$ which passes to the quotient. One easily checks that the induced map on the quotient is a ring isomorphism using that $\epsilon^n \oplus \epsilon^m \cong \epsilon^{n+m}$ and $\epsilon^n \otimes \epsilon^m \cong \epsilon^{mn}$. As $\mathbb{Z}$ is free, there is a splitting $K(X) \cong \tilde{K}(X) \oplus \mathbb{Z}$.

There is an alternative definition of $\tilde{K}(X)$.

Definition 3.6 ([5, Prop. 2.1])**.** Define the equivalence relation $\sim$ by setting, for two vector bundles E, E', $E \sim E'$ if and only if there exist $n, m \in \mathbb{N}$ such that $E \oplus \epsilon^n \cong E' \oplus \epsilon^m$. Then, define $\tilde{K}(X) := \{E \text{ vector bundles over } X\}/\sim$.

One can check that $\oplus$ is well-defined on the equivalence classes and defines a group structure on $\{E \text{ vector bundles over } X\}/\sim$.

Proposition 3.7. *The definitions 3.4 and 3.6 are equivalent in the sense that both groups are isomorphic.*

Proof. First, it is important to notice that, in view of remark 3.2, any element of $K(X)$ can be written uniquely in the form $\tilde{E} - \epsilon^n$ for some positive integer n. Indeed, consider an arbitrary formal difference $E - E'$. Adding a bundle E'' such that $E' \oplus E''$ is trivial to both sides sets the difference into the desired form (as $E - E' \simeq_s E \oplus E'' - E' \oplus E''$). In particular, we can identify $K(X)$ with the set of formal differences $\{E - \epsilon^n \mid E \text{ vector bundle on } X, n \in \mathbb{N}\}$. By definition (remark 3.5 1.),

$$\ker(K(X) \to K(x_0)) = \{E - \epsilon^n \mid E|_{x_0} \cong \epsilon^n\}$$

i.e. the kernel consists precisely of those formal differences $E - \epsilon^n$ such that E is an n-dimensional bundle on the connected component containing x_0. Let E_{x_0} denote the fibre of x_0. One can now describe the isomorphism between the two definitions. Define

$$\varphi : K(X) \to \{E \text{ vector bundles over } X\}/\sim, \; E - \epsilon^n \mapsto [E].$$

The reader is left to check that this is well-defined and a group homomorphism. We claim that $\varphi|_{\ker(K(X)\to K(x_0))}$ is an isomorphism. It is surjective because given an arbitrary $[E] \in \{E \text{ vector bundles over } X\}/\sim$, the formal difference $E - \epsilon^{\dim(E_{x_0})}$ maps to $[E]$. The kernel of φ is given by $\{[\epsilon^m - \epsilon^n] \mid n, m \in \mathbb{N}\}$. However, observe that for dimension reasons

$$\{[\epsilon^m - \epsilon^n] \mid n, m \in \mathbb{N}\} \cap \ker(K(X) \to K(x_0)) = [\epsilon^n - \epsilon^n]$$

which is precisely the neutral element. Hence, $\varphi|_{\ker(K(X)\to K(x_0))}$ is also injective and both definitions are equivalent. □

We define multiplication on $\{E \text{ vector bundles over } X\}/\sim$ via φ. In view of the previous proposition, we will use both definitions interchangeably as we please, and as best suited to the situation.

3.2 Facts about K-theory ([5, Section 2.1, 2.2])

Having defined the main objects, let us discuss a few facts about K-theory that will come in handy when trying to understand H-space structures on spheres. These facts will be given without proof as this would go out of the scope of one talk in a seminar.

1. K-theory is a cohomology theory. In particular, it is homotopy invariant and:

Proposition 3.8. *[5, Prop. 2.9] If $A \subset X$ is a closed subspace, the inclusion and quotient maps $A \xrightarrow{i} X \xrightarrow{q} X/A$ induce an exact sequence $\tilde{K}(X/A) \xrightarrow{q^*} \tilde{K}(X) \xrightarrow{i^*} \tilde{K}(A)$.*

One can extend this sequence to the left using suspensions i.e. the sequence

$$\cdots \longrightarrow \tilde{K}(\Sigma X) \xrightarrow{\Sigma i^*} \tilde{K}(\Sigma A) \xrightarrow{\delta} \tilde{K}(X/A) \xrightarrow{q^*} \tilde{K}(X) \xrightarrow{i^*} \tilde{K}(A) \tag{1}$$

remains exact. The connecting map δ is constructed as follows. Consider the diagram

$$\begin{tikzcd}
A \arrow[r, "i_1"] & X \arrow[r, "i_2"] & X \cup CA \arrow[r, "i_3"] \arrow[d, "q_1"] \arrow[dr, "q_1'"] & (X \cup CA) \cup CX \arrow[r, "i_4"] \arrow[d, "q_2"] \arrow[dr, "q_2'"] & (X \cup CA \cup CX) \cup C(X \cup CA) \arrow[d, "q_3"] \\
 & & X/A & SA & SX
\end{tikzcd}$$

where all horizontal maps are inclusions, the vertical maps are the maps collapsing the newly attached cone and the diagonal maps collapse the other space (left). Observe that, at each step, we attach the cone of the subspace two steps back in the sequence. It is left to the reader to convince himself that the vertical and diagonal maps really are those collapsing maps, but let us give some intuition for the first one. Recall that $X \cup CA := X \sqcup CA/\langle (0, a) \sim i(a) \rangle$. Hence, when collapsing CA to a point, we in particular collapse $\{0\} \times A \subset CA$ which is identified with the subspace A of X. We are left with the quotient X/A. If the vertical maps would induce isomorphisms on the reduced K-groups, we would be done because then the following diagram (2) is commutative and we already know that $\tilde{K}(X/A) \xrightarrow{q_1^*} \tilde{K}(X) \xrightarrow{i_1^*} \tilde{K}(A)$ as well as $\tilde{K}(SA) \xrightarrow{q_1'^*} \tilde{K}(X \cup CA) \xrightarrow{i_2^*} \tilde{K}(X)$ are exact.

$$\begin{tikzcd}
 & \tilde{K}(SA) \arrow[r] \arrow[d, "\cong"] \arrow[dr, "q_1'^*"] & \tilde{K}(X/A) \arrow[r, "q_1^*"] \arrow[d, "\cong"] & \tilde{K}(X) \arrow[r, "i_1^*"] & \tilde{K}(A) \\
\cdots \arrow[r] & \tilde{K}((X \cup CA) \cup CX) \arrow[r, "i_3^*"] & \tilde{K}(X \cup CA) \arrow[ur, "i_2^*"] & &
\end{tikzcd} \tag{2}$$

Then, since the vertical maps are isomorphisms, we conclude that the above horizontal sequence is exact as well (as is the induced inclusion sequence).

It remains to see that the vertical maps do induce isomorphisms on the reduced K-groups. This is [5, Lemma 2.10], which states:

Proposition 3.9. *If $B \subset X$ is a closed contractible subspace, then the quotient map $q : X \to X/B$ induces an isomorphism on the isomorphism classes of n-dimensionsal vector bundles. In particular, it induces an isomorphism on the reduced K-groups.*

Recall that the cone of a space is always contractible. Furthermore, observe that this proposition states that $\tilde{K}(SX) \cong \tilde{K}(\Sigma X)$ i.e. one may switch between reduced and unreduced suspension as we please.

2. We would like to calculate the values of the K-groups on certain spaces. In remark 3.5 3., it was shown that $K(\star) \cong \mathbb{Z}$, but this is all we know for now. Let us understand $K(S^2)$. Recall the canonical line bundle H over S^2 from example 2.4. We will get:

Proposition 3.10 ([5, Cor. 2.3])**.** *$K(S^2) \cong \mathbb{Z}[H]/((H-1)^2)$ and $\tilde{K}(S^2) = \langle H-1 \rangle$. Hence, as a group, $\tilde{K}(S^2) \cong \mathbb{Z}$, but its multiplication is completely trivial since $(H-1)^2 = 0$.*

Proof, sketch. The following is not a rigorous proof, rather some intuition why this could be true. One can show that the canonical line bundle over S^2 satisfies $(H \otimes H) \oplus \epsilon^1 \cong H \oplus H$ ([5, Example 1.13]). Algebraically, this relation rewrites as $(H-1)^2 = 0$ and one gets an injection $\mathbb{Z}[H]/((H-1)^2) \to K(S^2)$. It turns out that this is an isomorphism (see [5, Section 2.1]).

To calculate $\tilde{K}(S^2)$ we have to understand, for some fixed base point x_0 of S^2, the kernel of $i^* : K(S^2) \cong \mathbb{Z}[H]/((H-1)^2) \to \mathbb{Z} \cong K(x_0)$. Since H is a line bundle, $i^*(H) = 1$ and thus $\ker(i^*) = \langle H-1 \rangle$. $\square$

The fact that multiplication in $\tilde{K}(S^2)$ is completely trivial will be very important in many of the future arguments. Observe that this behaviour is very similar to singular cohomology.

3. For two compact CW-complexes X, Y, one can define an external product

$$\mu : K(X) \otimes K(Y) \to K(X \times Y),\ a \otimes b \mapsto p_1^*(a)p_2^*(b) := a \star b.$$

Here $p_1 : X \times Y \to X$ is projection onto the first coordinate and p_2 is projection onto the second. Similarly, one defines a reduced version

$$\tilde{\mu} : \tilde{K}(X) \otimes \tilde{K}(Y) \to \tilde{K}(X \wedge Y).$$

One can show that μ is an isomorphism if and only if $\tilde{\mu}$ is. It turns out that this map is very nicely behaved when $Y = S^2$ in the following sense:

Theorem 3.11 ([5, Thm. 2.2, 2.11], Bott periodicity)**.** *For any compact, Hausdorff space X, the maps*

$$\mu : K(X) \otimes K(S^2) \to K(X \times S^2)$$

and

$$\tilde{\mu} : \tilde{K}(X) \otimes \tilde{K}(S^2) \to \tilde{K}(X \wedge S^2)$$

are isomorphisms. In particular, since $\tilde{K}(S^2) \cong \mathbb{Z}$ and $X \wedge S^2 \cong \Sigma^2 X$, $\tilde{\mu}$ yields an isomorphism

$$\tilde{K}(X) \cong \tilde{K}(\Sigma^2 X),\ a \mapsto a \star H - 1.$$

This last isomorphism is the usual statement of Bott periodicity. Observe that this theorem turns the long exact sequence (1) into a six term circle. Hence, one can extend the sequence to the right using periodicity.

Specialising to the case when X is some sphere, observing that $S^n \wedge S^k \cong S^{n+k}$ and recalling from example 2.4 3) that $\tilde{K}(S^1) = 0$ as there are no non-trivial complex vector bundles over the circle, Bott periodicity gives:

Corollary 3.12. $\tilde{K}(S^n) = \begin{cases} 0 & \textit{if } n \textit{ is odd} \\ \mathbb{Z} & \textit{else.} \end{cases}$

Furthermore, from the explicit description of the Bott periodicity isomorphism, it is clear that $\tilde{K}(S^{2k}) = \langle (H-1) \star \cdots \star (H-1) \rangle$, the k-fold product of the generator of $\tilde{K}(S^2)$.

Moreover, the isomorphisms generalise as follows:

Corollary 3.13 ([5, Cor. 2.12])**.** *For any $k > 0$, the maps $K(X) \otimes K(S^{2k}) \to K(X \times S^{2k})$ and $\tilde{K}(X) \otimes \tilde{K}(S^{2k}) \to \tilde{K}(X \wedge S^{2k})$ are isomorphisms.*

Proof, sketch. This is simply iteration of the initial isomorphism. We can reason by induction, the case $k = 1$ being Bott periodicity. Suppose we have shown the statement for some k, let us show it also holds for $k + 1$. We have:

$$\begin{aligned}
\tilde{K}(S^{2k+2} \wedge X) &= \tilde{K}(S^2 \wedge S^{2k} \wedge X) \\
&\cong \tilde{K}(S^2) \otimes \tilde{K}(S^{2k} \wedge X) \\
&\cong \tilde{K}(S^2) \otimes \tilde{K}(S^{2k}) \otimes \tilde{K}(X) \\
&\cong \tilde{K}(S^{2k+1}) \otimes \tilde{K}(X)
\end{aligned}$$

where the second to last isomorphism is the induction hypothesis and the last isomorphism is Bott periodicity applied to the first two terms.

The unreduced statement follows since the two versions are equivalent.

□

In particular, one gets the following descriptions for the K-rings of spheres which will be needed in the next section.

Corollary 3.14. *For $k > 0$,*

$$K(S^{2k}) = \tilde{K}(S^{2k}) \oplus \mathbb{Z} \cong \mathbb{Z}[\alpha]/(\alpha^2)$$

where $\alpha = (H-1) \star \cdots \star (H-1)$ the k-fold external product of $H - 1$ and

$$K(S^{2k} \times S^{2k}) \cong \mathbb{Z}[\alpha, \beta]/(\alpha^2, \beta^2).$$

4 Back to H-space structures and introduction of the Hopf invariant

Having introduced this tool kit from K-theory, we are equipped with all the prerequisites needed to understand when spheres have an H-space structure. Remember from the introduction that, even though we have been a little distracted by K-theory, this is still our goal.

4.1 H-space Structure of Spheres

At this point, it is not too difficult to show that, when n is odd different from 1 i.e. $n - 1$ is even, S^{n-1} cannot admit an H-space structure. This is the content of:

Proposition 4.1 ([5, p. 60])**.** *Let $k > 0$ be some integer. There cannot be an H-space structure on S^{2k}.*

Proof. The proof goes by contradiction. Suppose there existed some H-space multiplication $\mu : S^{2k} \times S^{2k} \to S^{2k}$. Let e denote the left and right unit for μ. Recall from corollary 3.14 that

$$K(S^{2k}) \cong \mathbb{Z}[\gamma]/(\gamma^2) \text{ and } K(S^{2k} \times S^{2k}) \cong \mathbb{Z}[\alpha, \beta]/(\alpha^2, \beta^2)$$

where α is associated to the first (left) sphere and β to the second one. Consider the induced map $\mu^* : \mathbb{Z}[\gamma]/(\gamma^2) \to \mathbb{Z}[\alpha, \beta]/(\alpha^2, \beta^2)$. The aim is to understand $\mu^*(\gamma)$ and to derive a contradiction from this.

Introduce the inclusions into the first, respectively second, coordinate

$$i_1 : S^{2k} \to S^{2k} \times S^{2k},\ x \mapsto (x,e) \text{ and } i_2 : S^{2k} \to S^{2k} \times S^{2k},\ y \mapsto (e,y).$$

By construction, the compositions $\mu \circ i_1$ and $\mu \circ i_2$ are homotopic to the identity. In particular, it holds that $i_1^* \circ \mu^* = \mathrm{Id}^*$, similarly for i_2. Moreover, $i_1^*(\alpha) = \gamma$ and $i_1^*(\beta) = 0$. To see this, observe that $p_1 \circ i_1 = \mathrm{Id}$ where $p_1 : S^{2k} \times S^{2k} \to S^{2k}$ is the projection onto the first coordinate and $p_1 \circ i_2 = cst_e$. Considering the induced maps on K-theory and using that K-theory is homotopy invariant yields the desired result. Thus, $\gamma = i_1^* \circ \mu^*(\gamma)$ implies that $\mu^*(\gamma) = \alpha + n\beta + m\alpha\beta$ for n, m some integers i.e. the coefficient in front of α must be one. Repeating the same argument with i_2 yields $n = 1$ i.e. $\mu^*(\gamma) = \alpha + \beta + m\alpha\beta$. This yields a contradiction because since $\gamma^2 = \alpha^2 = \beta^2 = 0$,

$$0 = \mu^*(\gamma^2) = (\alpha + \beta + m\alpha\beta)^2 = 2\alpha\beta \neq 0.$$

□

Remark 4.2.
1. This proposition can also be shown using tools from singular cohomology, see for example [6, Exo 9, p. 143]. However, the calculations are arguably a bit more involved. It was chosen here to present a proof using K-theory since it will be needed anyways in section 5 and it is a nice way to assimilate the newly learned theory.
2. The condition on the sphere being even was used at the very beginning to ensure that the K-rings are actually of this form, in particular non zero.

4.2 The Hopf invariant

Understanding even values of n, i.e. odd values of $n-1$ turns out to be way more difficult. To do so, we will use a new tool: the Hopf invariant, giving their name to these notes. Let us begin by defining the Hopf invariant before explaining how this notion relates to H-space structures. As we have already excluded odd values of n, let us now always assume n to be even and write $n := 2m$. We will use both notations for n interchangeably in all that follows.

Let $f : S^{2n-1} \to S^n$ be some map. Consider $C_f = S^n \sqcup_f D^{2n}$ the mapping cone of f and observe that $C_f / S^n \cong S^{2n}$. One can now look at the corresponding long exact sequence (1) extended to the right by Bott periodicity (Thm 3.11):

$$\cdots \longrightarrow \tilde{K}(\Sigma S^{2m}) = 0 \longrightarrow \tilde{K}(S^{4m}) \xrightarrow{q^*} \tilde{K}(C_f) \xrightarrow{i^*} \tilde{K}(S^{2m}) \longrightarrow \tilde{K}(\Sigma S^{4m}) = 0$$

Let x be a generator of $\tilde{K}(S^{4m})$ and y a generator of $\tilde{K}(S^{2m})$. Let $\alpha = q^*(x)$ and β be some choice of preimage under i^* of y i.e.:

$$\begin{array}{c} \cdots \longrightarrow \tilde{K}(\Sigma S^{2m}) = 0 \longrightarrow \tilde{K}(S^{4m}) \xrightarrow{q^*} \tilde{K}(C_f) \xrightarrow{i^*} \tilde{K}(S^{2m}) \longrightarrow \tilde{K}(\Sigma S^{4m}) = 0 \\ x \longrightarrow \alpha \\ \beta \longrightarrow y \end{array} \tag{3}$$

Observe that $\beta^2 \in \ker q^*$ since $q^*(\beta^2) = y^2 = 0$ as multiplication in $\tilde{K}(S^{2m})$ is trivial. By exactness, $\ker q^* = \operatorname{im} i^* = \langle \alpha \rangle$. Hence, there exists some integer $h(f)$ such that $\beta^2 = h(f)\alpha$. This number $h(f)$ is the Hopf invariant of f. To summarise:

Definition 4.3 ([5, p.61])**.** Given $f : S^{2n-1} \to S^n$, the Hopf invariant of f denoted $h(f)$ is an integer such that the equality $\beta^2 = h(f)\alpha$ is satisfied, where β, α are as defined above (in (3)).

Remark 4.4. 1. Why is the Hopf invariant well-defined i.e. independent of the choice of β? Let β' be another element mapping to y under q^*. Then, β and β' differ by an element in the kernel, i.e. a multiple of α. Write $\beta' = \beta + n\alpha$ i.e. $\beta'^2 = \beta^2 + 2n\alpha\beta$. We claim that $\alpha\beta = 0$ which will conclude the proof of well-definedness. As $\alpha\beta \in \ker q^*$, there exists some integer k such that $\alpha\beta = k\alpha$. Multiplying with β on both sides: $0 = h(f)\alpha^2 = k\alpha\beta$ and multiplying with k on both sides yields $0 = k\alpha\beta = k^2\alpha$. This implies $k = 0$ by injectivity of i^* ($0 = i^*(k^2x) \implies k^2x = 0 \implies k = 0$ as x generates an infinite cyclic group).

2. We could have defined the Hopf invariant for odd n similarly, but this time $\tilde{K}(\Sigma S^n) \neq 0$ and $\tilde{K}(S^n) = 0$, so any element in $\tilde{K}(C_f)$ is a valid choice for β. Moreover, $q^*(\beta) = 0$, hence $\beta = k\alpha$ for some integer k and $\beta^2 = k^2\alpha^2$ as $\alpha^2 = i^*(x^2) = 0$. Thus, the Hopf invariant for odd n is always zero and therefore uninteresting.

3. Some readers might know the following definition of Hopf invariant in terms of singular cohomology: the Hopf invariant of f denoted $H(f)$ is the integer such that there is an isomorphism $H^*(C_f, \mathbb{Z}) \cong \mathbb{Z}[\alpha, \beta]/(\alpha^2, \alpha\beta, \beta^2 - H(f)\alpha)$. One can show that these two definitions are equivalent using Chern characters (see [6, Chapter 24.4]).

So why do we care about the Hopf invariant? How is this related to H-space structures on spheres? The following proposition sheds light on these questions.

Proposition 4.5 ([5, Lemma 2.18])**.** *For every H-space structure $\mu : S^{n-1} \times S^{n-1} \to S^{n-1}$, there exists an associated map $f_\mu : S^{2n-1} \to S^n$ which has Hopf invariant 1.*

Proof, sketch. We will give the construction of the associated map f_μ and explain what one should check to show it has Hopf invariant one, but we will not prove this explicitly. The remaining details can be found in the reference. The key observation to construct f_μ is that

$$S^{4m-1} = \partial D^{4m} = \partial(D^{2m} \times D^{2m}) = \partial D^{2m} \times D^{2m} \cup D^{2m} \times \partial D^{2m}.$$

This is helpful because $\partial D^{2m} \cong S^{2m-1}$ on which we have a map. It is also helpful to view $S^{2m} = D^{2m}_+ \cup D^{2m}_-$ as the union of the northern and southern hemispheres. Define:

$$\begin{array}{llllll} f_\mu : & \partial D^{2m} \times D^{2m} \;\cup & D^{2m} \times \partial D^{2m} & \longrightarrow & D^{2m}_+ \;\cup & D^{2m}_- \\ & (x,y) & & \longmapsto & |y|\mu(x, \frac{y}{|y|}) & \\ & & (x',y') & \longmapsto & & |x'|\mu(\frac{x'}{|x'|}, y') \end{array}$$

One can check that its restriction to $S^{n-1} \times S^{n-1}$ is just the multiplication, that this is continuous and always well-defined. To see that f_μ is of Hopf invariant one, we have to check that $\beta^2 = \alpha$ for the associated α, β as in (3). To do this, we introduce the notation $K(X, A) := \tilde{K}(X/A)$

and consider the following commutative diagram:

$$\begin{tikzcd}
\beta \otimes \beta \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & \beta^2 & \alpha \\
& \tilde{K}(C_f) \otimes \tilde{K}(C_f) \arrow[r] \arrow[d, "\cong"'] & \tilde{K}(C_f) & & \\
& K(C_f, D^{2m}_+) \otimes K(C_f, D^{2m}_-) \arrow[r] & K(C_f, S^{2m}) \cong \tilde{K}(S^{4m}) \arrow[u, "q^*"'] & & \\
\beta' \otimes \beta' \arrow[rrr, mapsto] & & & ? \arrow[uuu, mapsto] & x \arrow[uuu, mapsto]
\end{tikzcd}$$

where the horizontal maps are given by ring multiplication. The vertical map on the left is an isomorphism because the disks are contractible. We want the composition of the three maps to map $\beta \otimes \beta$ to α. Then, we could conclude by commutativity that $\beta^2 = \alpha$ as needed. To do this, all we need is for the lower horizontal map to map a generator to a generator (x) i.e. to be an isomorphism as we already know that x maps to α under q^*. This is the part of the proof that we leave to the reader. A hint is to express the lower horizontal map as composition of other maps which we know to be isomorphisms either by Bott periodicity or by construction of f_μ and unitality of μ (in particular one might want to consider the characteristic map of the $4m$-cell of C_f). $\square$

Remark 4.6. The converse to this theorem also holds. Whenever there is an element of Hopf invariant 1 in $\pi_{2n-1}(S^n)$, there is an associated H-space structure on S^{n-1}, see [4, Prop. 6.1.1].

5 Adams Operations and a Proof of the main Theorem

Recall that we still want to prove Theorem 1.1, i.e. that one cannot find a division algebra structure on $\mathbb{R}^n$ unless $n = 1, 2, 4, 8$. To do this, we should try to understand for which n the sphere S^{n-1} has an H-space structure. In the previous section (proposition 4.1), we excluded all odd values of n except 1 and, in view of proposition 4.5, realised that it is sufficient to understand for which n there exists or rather there doesn't exist an element of Hopf invariant 1 in $\pi_{2n-1}(S^n)$. In other words, we have reduced to showing the following theorem:

Theorem 5.1 ([5, Thm 2.19])**.** *Unless $n = 1, 2, 4, 8$, there is no element of Hopf invariant 1 in $\pi_{2n-1}(S^n)$.*

We will show this theorem by defining certain operations, the Adams operations, that will impose strong enough constraints on the possible values of n.

Theorem 5.2 ([5, Thm 2.20])**.** *For every integer $k \geq 0$, there exists a natural transformation*

$$\psi^k : K(-) \to K(-),$$

called the kth Adams operation, satisfying the following properties:

1. *$\psi^k_X : K(X) \to K(X)$ is a ring homomorphism.*
2. *For a line bundle L, $\psi^k(L) = L^k$.*
3. *For any integers $k, l \geq 0$, $\psi^k \circ \psi^l = \psi^{kl} = \psi^l \circ \psi^k$.*
4. *For any vector bundle E and any prime number p, $\psi^p(E) \equiv E^p \pmod p$.*

Remark 5.3. By naturality, ψ^k restricts to a well-defined operation on the reduced K-groups i.e. it maps kernel to kernel. Recalling the description of the reduced K-group as equivalence classes of vector bundles as in definition 3.6, the notation $\psi^k(E)$ is to be understood as the restriction of ψ^k to the reduced K-group applied to the equivalence class of E, some vector bundle (for example in properties 2 and 4 of the previous theorem).

Remark 5.4. A natural transformation between cohomology theories is called a cohomology operation. Hence, the Adams operations are one example of cohomology operations. The Steenrod squares give another example, this time for singular cohomology with mod 2 coefficients.

We will postpone the proof of theorem 5.2 until subsection 5.2. Let us rather get a better understanding of these operations by understanding how they act on spheres before finally proving theorem 5.1.

Lemma 5.5 ([5, Prop. 2.21])**.** *The map $\psi^k : \tilde{K}(S^{2m}) \cong \mathbb{Z} \to \tilde{K}(S^{2m}) \cong \mathbb{Z}$ is multiplication by k^m.*

Proof. First, consider the case $m = 1$. Since ψ^k is a ring homomorphism it is sufficient to understand what it does to $\gamma = H - 1$ the generator of $\tilde{K}(S^2)$. One calculates:

$$\begin{aligned}
\psi^k(\gamma) &= \psi^k(H) - \psi^k(1) && \text{additivity} \\
&= H^k - 1 && H \text{ and } 1 \text{ are line bundles and property 2} \\
&= (\gamma + 1)^k - 1 && \\
&= k\gamma + 1 - 1 = k\gamma && \text{multiplication is trivial}
\end{aligned}$$

as desired.
Then, one reasons by induction. Suppose the statement of the proposition to hold for some m and show it still holds for $m + 1$. Recall from theorem 3.11 and the following corollaries that $\tilde{K}(S^{2m+2}) = \langle \underbrace{(H-1) \star \cdots \star (H-1)}_{\text{m}} \star (H-1) \rangle$ and that $\underbrace{(H-1) \star \cdots \star (H-1)}_{\text{m}} := \delta$ is a generator of $\tilde{K}(S^{2m})$. In other words, the generator of $\tilde{K}(S^{2m+2})$ on which it suffices to calculate ψ^k is of the form $\delta \star \gamma = p_1^*(\delta)p_2^*(\gamma)$. One calculates:

$$\begin{aligned}
\psi^k(\delta \star \gamma) &= p_1^*\psi^k(\delta)p_2^*\psi^k(\gamma) && \text{naturality} \\
&= p_1^*(k^m\delta)p_2^*(k\gamma) && \text{induction hypothesis} \\
&= k^{m+1}p_1^*(\delta)p_2^*(\gamma) && p_i^* \text{ is a ring homomorphism} \\
&= k^{m+1}\delta \star \gamma &&
\end{aligned}$$

as desired. □

5.1 Proof of the Hopf Invariant One Theorem

The time has finally come to prove theorem 5.1.

Proof of Thm 5.1. Recall the definitions of β and α from (3). The proof strategy will be to understand what conditions the commutativity relations of the Adams operations (property 3) impose on β. Let us begin by calculating $\psi^k(\alpha)$ and $\psi^k(\beta)$. The first is easy to calculate using naturality and lemma 5.5:

$$\psi^k(\alpha) = \psi^k(q^*(x)) = q^*(\psi^k(x)) = q^*(k^{2m}x) = k^{2m}\alpha. \tag{4}$$

Similarly:

$$i^*(\psi^k(\beta) = \psi^k(y) = k^m y = i^*(k^m\beta).$$

Hence, $\psi^k(\beta)$ and $k^m\beta$ differ by some multiple of α. We write

$$\psi^k(\beta) = k^m\beta + \mu_k\alpha. \tag{5}$$

Let us now calculate the effect of the composition $\psi^k\psi^l(\beta)$ for two positive integers l and k.

$$\begin{aligned}
\psi^l\psi^k(\beta) &= \psi^l(k^m\beta + \mu_k\alpha) && \text{by (5)} \\
&= k^m\psi^l(\beta) + \mu_k\psi^l(\alpha) && \text{ring map} \\
&= k^m l^m\beta + k^m\mu_l\alpha + \mu_k l^{2m}\alpha && \text{by (4) and (5)} \\
&= k^m l^m\beta + (\mu_l k^m + \mu_k l^{2m})\alpha.
\end{aligned}$$

Analogously, one gets

$$\psi^k\psi^l(\beta) = k^m l^m\beta + (\mu_k l^m + \mu_l k^{2m})\alpha.$$

Since, by theorem 5.2 3, $\psi^k\psi^l = \psi^l\psi^k$, one concludes that $\mu_k l^m + \mu_l k^{2m} = \mu_l k^m + \mu_k l^{2m}$ or equivalently

$$\mu_l k^m(k^m - 1) = \mu_k l^m(l^m - 1). \tag{6}$$

We now specialise to $k = 2$ and $l = 3$. Observe that if $h(f) = 1$, μ_2 must be odd by theorem 5.2 4. Indeed, $2^m\beta + \mu_2\alpha = \psi^2(\beta) \equiv \beta^2 = h(f)\alpha = \alpha \pmod 2$ which implies $\mu_2 \equiv 1 \pmod 2$. Then (6) yields

$$\mu_3 2^m(2^m - 1) = \mu_2 3^m(3^m - 1).$$

In particular, 2^m must divide the right hand side but we have just shown that it cannot divide neither μ_2 nor 3^m. Hence, it must hold that $2^m \mid 3^m - 1$. It is an easy fact from number theory ([5, Lemma 2.22]) that this can only happen if $m = 1, 2, 4$ i.e. $n = 2, 4, 8$. □

So in the end, the existence of real division algebra structures on $\mathbb{R}^n$ boils down to the simple question of when 2^m divides $3^m - 1$. This seems nearly a bit disappointing. The actual mathematical richness, however, comes from our journey to here. The first enlightening observation was to change the algebraic statement into a question about H-space structures on spheres (see section 1) and later into a question of existence of elements of Hopf invariant one (section 4). The main key was to find a well-adapted cohomology theory, namely K-theory, on which one could construct operations (the Adams operations) whose relations are restrictive enough to exclude all other values of n.

Maybe the reader was already aware of a similar proof, using Steenrod squares, to show that the only values of n that admitted an element of Hopf invariant 1 were powers of 2. Indeed, using the definition of Hopf invariant in terms of singular cohomology (see remark 4.4 3), one can show that there is $f \in \pi_{2n-1}(S^n)$ of Hopf invariant 1 if and only if the nth Steenrod square $Sq^n : H^n(C_f, \mathbb{Z}/2\mathbb{Z}) \to H^{2n}(C_f, \mathbb{Z}/2\mathbb{Z})$ is non zero. The Adem relations prove that whenever n is not a power of 2, Sq^n factors through a zero cohomology group and thus is zero. However, this method cannot say anything about the powers of 2. In his original proof in [1], Adams forces new relations on the Steenrod squares, so called secondary cohomology operations, by restricting the domain to some subset of elements vanishing for certain Steenrod squares and making the target into a quotient of the cohomology group by certain Adem relations. He then, except for the values $2, 4, 8$, finds relations between the power two Steenrod squares up to some equivalence class. In other words, he finds a way to write $Sq^{2^i} = \Sigma a_k\Phi_k \pmod{\Sigma b_i\psi_i}$ where the a_k, b_i are Steenrod squares and Φ_k, ψ_i some secondary cohomology operations and observes that the expression on the right lands in a cohomology group that is zero (i.e. one has $Sq^{2^n} = 0 \pmod 0$ which yields what we wanted). Adams' idea is not really more complicated then the proof that was presented here. However, to show that everything holds rigorously is tedious work that was carefully done in his 80 pages paper [1].

5.2 Construction of the Adams Operations

To finish off nicely, let us explain how to construct the Adams operations i.e. prove theorem 5.2. To get an idea of how to construct them, one should understand how ψ^k acts on a vector bundle E which is some direct sum of line bundles $L_1 \oplus \cdots \oplus L_n$. Additivity and the definition on line bundles yield:

$$\psi^k(E) = L_1^k + \cdots + L_n^k.$$

An important observation to make here is that this is a symmetric polynomial of degree k in n variables.

The second key observation involves exterior powers which we recall from definition 2.7. The i^{th} exterior power of a vector bundle E, $\lambda^i(E)$ behaves very much like the exterior power of a vector space. In particular:

Lemma 5.6 ([5, p. 62])**.** *Let E, E_1, E_2 be any vector bundles. The exterior power operation has the following properties:*

1. $\lambda^i(E_1 \oplus E_2) \cong \oplus_{k=0}^i \lambda^k(E_1) \otimes \lambda^{i-k}(E_2)$.
2. $\lambda^0(E) = \epsilon^1$ *the trivial line bundle,* $\lambda^1(E) = E$.
3. $\lambda^i(E) = 0$ *whenever i is greater than the maximum dimension of the fibers of E.*

The proof is left as an exercise to the reader/these properties are believed to be well-known at this point of the studies. Let us go back to considering the vector bundle E which is a direct sum of n line bundles. From lemma 5.6, it follows that

Lemma 5.7 ([5, p. 62-63])**.** *If $E = L_1 \oplus \cdots \oplus L_n$, $\lambda^i(E) = \sigma_i(L_1, \ldots, L_n)$ where σ_i is the i^{th} elementary symmetric polynomial, the sum of all i distinct products of the L_j.*

Let us look at the following example to convince ourselves of this fact, (this is of course not a proof, but illustrates how the proof would go).

Example 5.8. Let $E = L_1 \oplus L_2 \oplus L_3$. Observe that as L_i is a line bundle, by lemma 5.6 3 $\lambda^2(L_i) = 0$. Calculate that $\lambda^2(E) = L_1 \otimes L_2 \oplus L_1 \otimes L_3 \oplus L_3 \otimes L_2 = \sigma_2(L_1, L_2, L_3)$:

$$\begin{aligned}
&\lambda^2(L_1 \oplus L_2 \oplus L_3) \\
&= (\lambda^0(L_1) \otimes \lambda^2(L_2 \oplus L_3)) \oplus (\lambda^1(L_1) \otimes \lambda^1(L_2 \oplus L_3)) \oplus (\lambda^2(L_1) \otimes \lambda^0(L_2 \oplus L_3)) && \text{lemma 5.6 1} \\
&= \lambda^2(L_2 \oplus L_3) \oplus (L_1 \otimes \lambda^1(L_2 \oplus L_3)) && \text{lemma 5.6 2, 3} \\
&= (\lambda^1(L_2) \otimes \lambda^1(L_3)) \oplus (L_1 \otimes (\lambda^1(L_2) \otimes \lambda^0(L_3) \oplus \lambda^0(L_2) \otimes \lambda^1(L_3))) && \text{5.6 1 (and 2,3) again} \\
&= (L_2 \otimes L_3) \oplus (L_1 \otimes L_2) \oplus (L_1 \otimes L_3).
\end{aligned}$$

This property is extremely nice because of the following well-known theorem:

Theorem 5.9 (Fundamental Theorem of Symmetric Polynomials)**.** *For any degree k symmetric polynomial $p(t_1, \ldots, t_n)$, there exists a unique polynomial (not depending on n) $s_k(x_1, \cdots, x_k)$ such that $s(\sigma_1(t_1, \ldots, t_n), \ldots, \sigma_k(t_1, \ldots, t_n)) = p(t_1, \ldots, t_n)$.*

Denote by s_k the polynomial associated to $p(L_1, \ldots, L_n) = L_1^k + \cdots + L_n^k$. Then, observe that, for the vector bundle E from above, setting $\psi^k(E) = s_k(\lambda^1(E), \ldots, \lambda^k(E))$ has exactly the properties required in theorem 5.2. This motivates:

Theorem 5.10 ([5, p. 62-63])**.** *For a vector bundle E, defining $\psi^k(E) = s_k(\lambda^1(E), \ldots, \lambda^k(E))$, with s_k being the polynomial associated to $t_1^k + \cdots + t_n^k$ by the fundamental theorem on symmetric polynomials and extending this construction to $K(X)$ by setting $\psi^k(E - E') = \psi^k(E) - \psi^k(E')$ fulfills the requirements of theorem 5.2.*

To prove this theorem, the following black box will be needed. This makes the link between general vector bundles and line bundles.

Lemma 5.11 (Splitting Principle)**.** *For any vector bundle $\pi : E \to X$, there exists a compact, Hausdorff space $F(E)$ and a continuous map $p' : F(E) \to X$ such that the induced map $p'^* : K(X) \to K(F(E))$ is injective and $p^*(E)$ is a sum of line bundles.*

For a proof, see the section in [5] entitled "the splitting principle".

Proof of Thm 5.10, sketch. Naturality of the construction is straightforward from naturality of exterior powers (i.e. the fact that exterior powers and pullbacks commute) and the fact that ring homomorphisms commute with polynomials.
Let us now prove additivity. The strategy is to use the splitting principle to reduce to the line bundle case where it is easy to check. All the other required properties follow by extremely similar arguments and their proofs will hence be omitted here.
Let E_1, E_2 be two vector bundles over X. Let p_1 be the map given by the splitting principle for E_1, such that $p_1^*(E_1) = L_1 \oplus \cdots \oplus L_{n_1}$ for certain line bundles L_i. Similarly, let p_2 be the map given by the splitting principle for E_2, such that $p_2^*(E_2) = L'_1 \oplus \cdots \oplus L'_{n_2}$ for certain line bundles L'_i. Then:

$$\begin{aligned}
&(p_1 \oplus p_2)^*(\psi^k(E_1 \oplus E_2)) && \\
&= \psi^k((p_1 \oplus p_2)^*(E_1 \oplus E_2)) && \text{by naturality} \\
&= \psi^k(L_1 \oplus \cdots \oplus L_{n_1} \oplus L'_1 \oplus \cdots \oplus L'_{n_2}) && \text{by definition of } p_i \\
&= L_1^k + \cdots + L_{n_1}^k + L_1'^k + \cdots + L_{n_2}'^k && \text{by construction} \\
&= \psi^k(L_1 \oplus \cdots \oplus L_{n_1}) \oplus \psi^k(L'_1 \oplus \cdots \oplus L'_{n_2}) && \text{by construction} \\
&= (p_1 \oplus p_2)^*(\psi^k(E_1) \oplus \psi^k(E_2)) && \text{definition of } p_i \text{ and naturality.}
\end{aligned}$$

One then concludes by injectivity of $(p_1 \oplus p_2)^*$ that $\psi^k(E_1) \oplus \psi^k(E_2) = \psi^k(E_1) \oplus E_2)$ as wanted. ☐

Remark 5.12. From the construction, it is clear that asking for the Adams operations to be a ring homomorphism and the action on line bundles determine the Adams operations completely. The other properties follow.

References

[1] J. F. Adams. On the non-existence of elements of Hopf invariant one. *Ann. Math. (2)*, 72: 20–104, 1960. ISSN 0003-486X. doi: 10.2307/1970147. URL `www.jstor.org/pss/1970147`.

[2] J. F. Adams and M. F. Atiyah. K-theory and the Hopf invariant. *Q. J. Math., Oxf. II. Ser.*, 17:31–38, 1966. ISSN 0033-5606. doi: 10.1093/qmath/17.1.31.

[3] M. F. Atiyah. *K-theory. With notes by D. W. Anderson.* Redwood City, CA etc.: Addison-Wesley, reprint edition, 1989. ISBN 0-201-09394-4.

[4] J. R. Harper. *Secondary cohomology operations*, volume 49 of *Grad. Stud. Math.* Providence, RI: American Mathematical Society (AMS), 2002. ISBN 0-8218-3270-0.

[5] A. Hatcher. *Vector bundles and K-theory.* 2017. URL `https://pi.math.cornell.edu/~hatcher/VBKT/VB.pdf`.

[6] J. P. May. *A concise course in algebraic topology.* Chicago, IL: University of Chicago Press, 1999. ISBN 0-226-51182-0; 0-226-51183-9.

[7] M. Suarez-Alvarez, 2018. URL `https://mathoverflow.net/questions/327881/every-complex-vector-bundle-over-the-circle-is-trivial`.

[8] B. L. van der Waerden. Hamilton's discovery of quaternions. *Math. Mag.*, 49:227–234, 1976. ISSN 0025-570X. doi: 10.2307/2689449.