The Smash Product Theorem

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These are the notes to an expository talk on the smash product theorem through the lense of descent and nilpotence. I gave it at the European Talbot on chromatic homotopy theory in 2025.

Contents

1 Introduction

This talk's goal is to discuss and prove the *Smash Product Theorem* due to Hopkins and Ravenel:

Theorem 1.1 ([3, Thm. 7.5.6])**.** *Bousfield localisation at a Morava E-theory, L_{E_n}, is smashing. In other words, for any spectrum X, there is an equivalence $L_{E_n}X \simeq L_{E_n}\mathbb{S} \otimes X$ that is natural in the spectrum X.*

Notation 1.2. Throughout, we will freely switch between L_{E_n} and L_n to denote the localisation functor at some Morava E-theory of height n.

Observe that lax symmetric monoidality of L_n gives a map

$$L_n\mathbb{S} \otimes X \to L_n\mathbb{S} \otimes L_nX \to L_nX.$$

This is an E_n-equivalence since tensoring both sides with E_n gives

$$E_n \otimes L_n\mathbb{S} \otimes X \simeq E_n \otimes X \simeq L_nX \otimes E_n.$$

As an E_n-equivalence between E_n-local objects is an equivalence[1], the smash product theorem follows if one can show that $L_n\mathbb{S} \otimes X$ is E_n-local. This would be fine if we knew that $L_n\mathbb{S}$ is obtained from finite colimits (equivalently finite limits), retracts and shifts of E_n, as then $L_n\mathbb{S} \otimes X$ is obtained from finite limits, retracts and shifts of $E_n \otimes X$. Observe that $E_n \otimes X$ is a E_n-module and as such it is E_n-local. Then, $L_n\mathbb{S} \otimes X$ is also E_n-local because limits and

[1] The fiber of an E_n-equivalence between E_n-local objects is both E_n-acyclic and E_n-local as limit of local objects. Thus, it must be zero since id_F must be trivial.

retracts of local objects are local.

This is the approach taken here. The first third of this talk introduces the ideas of "nilpotence and descent" as exposed in the first sections of [2]. This provides the necessary language to formalise the idea of $L_n\mathbb{S}$ being obtained via finite colimits, retracts and shifts of E_n, and also relate this to a statement on a localisation being smashing. Then, we explain how to use this language to prove the Smash Product Theorem in Section 3. This is based on the exposition in modern language of [2, Sec. 3.5], with details for certain results from [1, Lect. 31] and [3, Ch. 8] (the original reference). Finally, we illustrate another application of this formalism by giving a slick proof of the Chromatic Convergence Theorem, which was shown to us by Ishan Levy.

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2 Descent and Nilpotence

Throughout this section, let $\mathcal{C}$ be a stably symmetric monoidal idempotent complete ∞-category and let A be an $\mathbb{E}_1$-algebra in $\mathcal{C}$. In our applications, $\mathcal{C}$ will be the category of spectra or E_n-local spectra and A will be E_n, some Morava E-theory of height n. However, the ideas discussed here are valuable in this more general setting.

We begin by recalling the following definition from a previous talk:

Definition 2.1 ([2, Def. 2.1])**.** A *thick subcategory* of $\mathcal{C}$ is a full subcategory $\mathcal{C}'$ such that $0 \in \mathcal{C}'$, if two out of three objects in a cofiber sequence are in $\mathcal{C}'$ then so is the third, and $\mathcal{C}'$ is idempotent complete, i.e. if $X \oplus Y \in \mathcal{C}'$, then so are both X and Y.
Given an object $A \in \mathcal{C}$, we write $\mathrm{Thick}(A)$ for the smallest thick subcategory containing A.

Definition 2.2 ([2, Def. 2.7])**.** An object X in $\mathcal{C}$ is *A-nilpotent* if it is in the *thick $\otimes$-ideal generated by A*, denoted $\mathrm{Thick}^{\otimes}(A)$ and defined as $\mathrm{Thick}(\{A \otimes Y\}_{Y \in \mathcal{C}})$. In other words, X is A-nilpotent if it belongs to the thick subcategory generated by all elements of the form $A \otimes Y$, $Y \in \mathcal{C}$. We also write Nil_A for $\mathrm{Thick}^{\otimes}(A)$.

Exercise 2.3. Show that Nil_A is an ideal, i.e. if $X \in \mathrm{Nil}_A$, $Y \in \mathcal{C}$ then $X \otimes Y \in \mathrm{Nil}_A$.

The following example justifies the terminology.

Example 2.4 ([2, Ex. 2.8])**.** Let $\mathcal{C}$ be $D(\mathbb{Z})$ the (derived) category of $\mathbb{Z}$-modules, and let $A := \mathbb{Z}/p\mathbb{Z}$ for some prime p. Then, an object X is A-nilpotent if and only if there exists $n > 0$ such that multiplication by $p^n : X \to X$ is null-homotopic.

As a first step towards proving the Smash Product Theorem, we would like to argue that $L_n\mathbb{S}$ is E_n-nilpotent, which, we remind the reader, implies that it consists of finite colimits, retracts and shifts of E_n-locals as desired in the introduction. To achieve this, it is useful to have a way of approximating a given object by nilpotent ones. That will be achieved by the *cobar construction.*

Definition 2.5 ([2, Const. 2.10])**.** The *augmented cobar construction of A*, $CB^{aug}(A)$ is the augmented cosimplicial diagram

$$1 \longrightarrow A \rightrightarrows A^{\otimes 2} \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} A^{\otimes 3} \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \cdots$$

where the maps are given by the unit, the various multiplications and inclusions into the factors of the tensor product. The *cobar construction* $CB^{\bullet}(A)$ is $CB^{aug}(A)|_{\Delta}$.

Remark 2.6. The augmented cobar construction $CB^{aug}(A)$ admits a splitting/extra degeneracy after tensoring with A coming from the multiplication. In particular, for any functor $F : \mathcal{C} \to \mathcal{D}$, the natural map $F(A) \to \mathrm{Tot}(F(CB^\bullet(A) \otimes A))$ is an equivalence.

We wanted tensoring with the cobar construction to be a way of approximating a given object by nilpotent ones. The following proposition shows that this succeeds for objects that were already nilpotent as they are recovered by the totalisation of the cobar construction.

Proposition 2.7 ([2, Prop. 2.12])**.** *Let $X \in \mathrm{Nil}_A$. For any stable ∞-category $\mathcal{D}$ and any exact functor $F : \mathcal{C} \to \mathcal{D}$, the natural map $F(X) \to \mathrm{Tot}(F(X \otimes CB^\bullet(A)))$ is an equivalence. In particular, X is the limit of $X \otimes CB^\bullet(A)$.*

Proof. The class for which the equivalence holds is thick. Thus, to get it for all A-nilpotents, it suffices to check on the generators $A \otimes Y$ for some $Y \in \mathcal{C}$. In that case, $X \otimes CB^{aug}(A)$ has an extra degeneracy. This implies that the desired map is an equivalence (see the previous remark). □

Recall that our aim still is to show that $L_n\mathbb{S}$ is E_n-nilpotent. How can one do that? The previous discussion suggests that one might try to find a criterion for $CB^\bullet(E_n) \otimes L_n\mathbb{S}$ that forces its totalisation to be $L_n\mathbb{S}$. The necessary language will be introduced now.

Definition 2.8 ([2, Def. 2.15])**.** A *tower* in $\mathcal{C}$ is a functor $\mathbb{Z}_{\geq 0}^{op} \to \mathcal{C}$. These are the objects of the stable ∞-category $\mathrm{Tow}(\mathcal{C}) \coloneqq \mathrm{Fun}(\mathbb{Z}_{\geq 0}^{op}, \mathcal{C})$.

Construction 2.9 ([2, Const. 2.16])**.** To any cosimplicial object $X^\bullet : \Delta \to \mathcal{C}$, one can associate a tower $\{\mathrm{Tot}_{\leq n}(X^\bullet)\}_{n \geq 0}$ via the sequence of partial totalisations, i.e. by defining the n^{th} object in the tower to be

$$\mathrm{Tot}_{\leq n}(X^\bullet) \coloneqq \lim_{[i] \in \Delta, i \leq n} X^i.$$

Observe that the limit of this tower is the totalisation of the cosimplicial object $X^\bullet$.

Fact 2.10. The cosimplicial object can be recovered from its tower, i.e. there is an equivalence of categories $\mathrm{Fun}(\Delta, \mathcal{C}) \simeq \mathrm{Tow}(\mathcal{C})$ (stable Dold-Kan correspondence).

In light of this fact, we will use $CB^\bullet(A)$ to denote both the cobar construction and the associated tower.

Definition 2.11 ([2, Def. 2.17, 2.19])**.** Let $\{X_i\}_{i \geq 0}$ be a tower in $\mathcal{C}$.

- It is *nilpotent* if there exists $N \in \mathbb{N}$ such that for all $i \in \mathbb{N}$ the natural map $X_{i+N} \to X_i$ is nullhomotopic.
- It is *quickly converging* if it is in the thick subcategory of $\mathrm{Tow}(\mathcal{C})$ generated by the constant and nilpotent towers.
- A cosimplicial object is *nilpotent*, respectively *quickly converging*, if its associated tower is nilpotent, respectively quickly converging.

In the situation of quick convergence, the totalisation, which is usually an infinite limit, behaves like a finite limit in many aspects:

Proposition 2.12 ([2, Prop. 2.20, 2.22])**.** *Let $X^\bullet \in Fun(\Delta, \mathcal{C})$ be quickly converging. Let $\mathcal{D}$ be any idempotent complete stable ∞-category and $F : \mathcal{C} \to \mathcal{D}$ an exact functor. Then:*

1. $\mathrm{Tot}(X^\bullet)$ *exists*[2].

[2] In our applications, $\mathcal{C}$ will always be assumed complete, so this statement is not relevant to us.

2. *$F(X^\bullet)$ is quickly converging in $\mathcal{D}$.*

3. *The natural map $F(\mathrm{Tot}(X^\bullet)) \to \mathrm{Tot}(F(X^\bullet))$ is an equivalence.*

4. *$\lim_i \mathrm{Tot}_{\le i}(X^\bullet) = \mathrm{Tot}(X^\bullet)$ is a retract of some $\mathrm{Tot}_{\le j}(X^\bullet)$ and $\mathrm{Tot}(X^\bullet) \in \mathrm{Thick}(X^i | i \in \mathbb{N})$*

Proof. As exact functors respect finite limits, one can replace the totalisation by the inverse limit of the partial totalisations and check the corresponding properties on the towers. Observe that the class of towers for which the first three properties hold is thick and clearly contains the constant and nilpotent towers.
For the last property, if $\{Y_i\}_{i\ge 0}$ is a quickly converging tower with limit Y, then, by a dual statement to 3. applied to $F := Hom_{\mathcal{C}}(-, Y)$, one gets $\mathrm{colim}_i\, Hom_{\mathcal{C}}(Y_i, Y) \simeq Hom_{\mathcal{C}}(Y, Y)$. The equivalence is given by precomposing with the natural map $\iota_i : Y \to Y_i$. In particular, there must exist some $f : Y_i \to Y$ such that $f \circ \iota_i \simeq id_Y$, i.e. the limit of the tower is a retract of some finite stage. For the second half of property 4., note that each partial totalisation $\mathrm{Tot}_{\le n}(X^\bullet)$ is in the thick subcategory generated by the X^i, $0 \le i \le n$. Since we just showed that the totalisation is a retract of a partial totalisation, this concludes the proof. □

As it turns out, quick convergence of the cobar tower is closely related to nilpotence.

Proposition 2.13 ([2, Prop. 2.26])**.** *An object X is A-nilpotent if and only if $CB^\bullet(A) \otimes X$ is quickly converging with limit X.*

Proof. For the proof, we will need the following two facts:

1. Let I denote the fiber of the unit map $\mathbb{1} \to A$. If X is A-nilpotent, there exists some $N \ge 0$ such that the map $I^{\otimes N} \to \mathbb{1}$ becomes null after tensoring with X.

2. There is a cofiber sequence of towers

$$\{X \otimes I^{\otimes n}\}_{n\ge 0} \to \mathrm{Const}(X) \to \{\mathrm{Tot}_{\le n}(CB^\bullet(A) \otimes X)\}_{n\ge 0}.$$

   Here $\mathrm{Const}(X)$ denotes the constant tower with value X.

Let us assume $X \in \mathrm{Nil}_A$. Then, by the first fact, there exists $N \ge 0$ such that $(I^{\otimes N} \to \mathbb{1}) \otimes X \simeq 0$. In other words, the tower $\{X \otimes I^{\otimes n}\}_{n\ge 0}$ is N-nilpotent. Thus, by the second fact, $CB^\bullet(A) \otimes X$ is the cofiber of a constant and a nilpotent tower and therefore quickly converging. As the limit of a nilpotent tower is zero, passing to the limit on the cofiber sequence gives that $CB^\bullet(A) \otimes X$ converges to X.
Conversely, if $CB^\bullet(A) \otimes X$ is quickly converging to X, by Property 4 of Proposition 2.12,

$$X = \mathrm{Tot}(CB^\bullet(A) \otimes X) \in \mathrm{Thick}(A \otimes X, A \otimes A \otimes X, \dots)$$

the latter category is contained in $\mathrm{Thick}^{\otimes}(A)$, which concludes the proof. □

Thus, we have reduced proving the Smash Product Theorem to the statement that the cobar construction of E_n is quickly converging (see Thm. 3.2) or, equivalently, that the unit of the L_n-local category is E_n-nilpotent. There is terminology to designate these particularly nice situations.

Definition 2.14 ([2, Def. 2.30])**.** We say that A is *descendable* in $\mathcal{C}$ if the unit of $\mathcal{C}$ is A-nilpotent.

Observe that in this case $\mathcal{C} = \mathrm{Thick}^{\otimes}(A)$. Hence, we aim to show that E_n is descendable in the L_n-local category.

Descendability is a strong and very useful condition. If an object A is descendable, many properties can be checked after tensoring with it or up to nilpotenece. For example, an object is null if and only if its tensor product with A is null ([2, Prop. 2.31]) or one can check if an element in the homotopy groups of a ring spectrum is nilpotent by checking if it maps to zero in the homotopy groups of that ring spectrum tensored with A ([2, Prop. 2.33]).
Moreover, the Devinatz-Hopkins-Smith nilpotence theorem can be shown by refining those methods of descendability. Indeed, MU is not descendable in spectra (there exist non-trivial spectra that tensored with MU are null, like the Brown-Comennetz dual of the sphere), but it is close enough to being descendable. This can be quantised by determining the asymptotic growth of some *exponent function* and deducing the nilpotence theorem; see [2, Sec. 3.4].
The language presented here is also the starting point to discussing higher categorical analogues to faithfully flat and Galois descent; see [2, Sec. 2.5, 3.1].

3 Hopkins-Ravenel Smash Product Theorem

In this section, we summarise the previous discussion to deduce the Smash Product Theorem (Thm. 1.1). We begin by assembling the ingredients of the last section to explain how quick convergence of an $\mathbb{E}_1$-ring spectrum relates to smashingness of the localisation in general. Then, we focus on showing that $CB^\bullet(E_n)$ is quickly converging.
The following proposition is essentially a recap of concepts we have already seen.

Proposition 3.1 ([2, Prop. 3.37]). *Let E be an $\mathbb{E}_1$-ring spectrum. Suppose $CB^\bullet(E)$ is quickly converging. Then,*

1. $\mathrm{Tot}(CB^\bullet(E)) = L_E\mathbb{S}$.
2. *$L_E\mathbb{S}$ is E-nilpotent.*
3. *$L_E : Sp \to Sp$ is a smashing localisation.*

Proof. 1. By quick convergence and as $E \otimes -$ is exact, applying Proposition 2.12 3., one gets

$$E \otimes \mathrm{Tot}(CB^\bullet(E)) \simeq \mathrm{Tot}(CB^\bullet(E) \otimes E) \simeq E.$$

The last equivalence can be deduced from Proposition 2.7 as E is E-nilpotent. Thus, the natural map $L_E\mathbb{S} \to \mathrm{Tot}(CB^\bullet(E))$ is an E-equivalence. Moreovoer, $\mathrm{Tot}(CB^\bullet(E))$ is E-local as a limit of E-local objects[3].

2. By Proposition 2.13, we need to show that $CB^\bullet(E) \otimes L_E\mathbb{S}$ is quickly converging with limit $L_E\mathbb{S}$. The following equivalences

$$CB^\bullet(E) \otimes L_E\mathbb{S} \simeq CB^\bullet(E \otimes L_E\mathbb{S}) \simeq CB^\bullet(E)$$

ensure quick convergence and imply that $\mathrm{Tot}(CB^\bullet(E) \otimes L_E\mathbb{S}) \simeq L_E\mathbb{S}$ using 1.

3. As explained in the introduction, it suffices to argue that $L_E\mathbb{S} \otimes X$ is E-local for all spectra X. By 2., $L_E\mathbb{S} \in \mathrm{Thick}^\otimes(E)$ and thus, as $\mathrm{Thick}^\otimes(E)$ is a tensor ideal, $L_E\mathbb{S} \otimes X \in \mathrm{Thick}^\otimes(E)$ as well. This implies it is E-local as the E-local objects form a thick subcategory containing the generators of $\mathrm{Thick}^\otimes(E)$.

□

Thus, it suffices to show the following:

[3] Each term of the cobar construction, $E^{\otimes i}$, is E-local as it is an E-module.

Theorem 3.2 ([2, Thm. 3.38], [1, Lect. 31]). *The cobar construction on Morava E-theory $CB^\bullet(E_n)$ is quickly converging.*

How would one go about proving such a statement? In the previous section, we explained that quick convergence is really a notion concerning the towers associated to the cosimplicial objects. The attentive reader might already suspect that quick convergence is reflected in the corresponding spectral sequences.

Construction 3.3 ([2, Ex. 2.11]). One can associate a spectral sequence (the Bousfield-Kan spectral sequence) to the tower corresponding to $CB^\bullet(A) \otimes X$ of the following signature:

$$E_2^{s,t} = H^s(\pi_t(CB^\bullet(A) \otimes X)) \Rightarrow \pi_{t-s}(\mathrm{Tot}(CB^\bullet(A) \otimes X)).$$

Proposition 3.4 ([2, Prop. 2.21]). *If $CB^\bullet(A) \otimes X$ is quickly converging, the associated BKSS has a horizontal vanishing line at some finite stage, i.e. there exists $N \geq 2$ and $h \geq 0$ such that $E_N^{s,t} = 0$ for all $s > h$.*

One can now wonder whether the converse also holds. Does the existence of a horizontal vanishing line imply quick convergence? This is unfortunately not always the case, but for the spectral sequence coming from the cobar construction of an $\mathbb{E}_1$-ring spectrum, one gets close enough:

Theorem 3.5. *Let E be an $\mathbb{E}_1$-ring spectrum. If there exists $s \geq 1$ such that for any finite spectrum F the BKSS for $\pi_*(\mathrm{Tot}(CB^\bullet(E) \otimes F))$ vanishes at $E_s^{p,q}$ for $p \geq s$, then $CB^\bullet(E)$ is quickly converging.*

Proof Idea. Let $\{Y_n\}_{n\geq 0}$ denote the tower with n^{th} term $\mathrm{cofib}(\mathrm{Tot}\, CB^\bullet(E) \to \mathrm{Tot}_{\leq s2^n} CB^\bullet(E))$ with the expected natural maps between the terms. One uses the horizontal vanishing line to show that all maps in the tower of graded abelian groups $\{\pi_*(Y_n)\}_{n\geq 0}$ are trivial. Since this remains true when tensoring with any finite spectrum, the zero maps on the homotopy groups remain zero maps on all homologies. In other words, they are phantom maps[4]. Then, one uses the fact that in spectra the composition of two phantom maps is null (see [1, Lect. 30, Lemma 5]) to deduce that the tower $\{Y_n\}_{n\geq 0}$ is 2-nilpotent. In particular, the tower $\{\mathrm{Tot}_{\leq s2^n} CB^\bullet(E))\}_{n\geq 0}$ sits in a cofiber sequence with a constant (the totalization) and a nilpotent one. It is quickly converging!
However, this is not enough to deduce that the entire tower, and not only this subtower, is quickly converging. This uses the explicit description of the cofiber in this case. We aim to use Fact 2. of Proposition 2.13 to show that $Y_n = I^{\otimes s2^n}[1]$ where $I := \mathrm{fib}(L_E\mathbb{S} \to E)$ and the maps are given by successive tensoring with (appropriate powers of) $I \to L_E\mathbb{S}$. This is clear if one argues that $\mathrm{Tot}(CB^\bullet(E)) = L_E\mathbb{S}$. Indeed, then

$$Y_n \simeq \mathrm{cofib}(\mathrm{Tot}\, CB^\bullet(E) \to \mathrm{Tot}_{\leq s2^n} CB^\bullet(E)) \simeq \mathrm{fib}(L_E\mathbb{S} \to \mathrm{Tot}_{\leq s2^n} CB^\bullet(E))[1] \simeq I^{\otimes s2^n}[1]$$

where the first equivalence is by construction and the last by Fact 2. of Proposition 2.13.
To show that $\mathrm{Tot}(CB^\bullet(E)) = L_E\mathbb{S}$, observe that

$$\mathrm{Tot}(CB^\bullet(E)) \simeq \lim_i \mathrm{Tot}_{\leq i}(CB^\bullet(E)) \simeq \lim_i \mathrm{Tot}_{\leq s2^i}(CB^\bullet(E)) \simeq \mathrm{Tot}(CB^{s2^\bullet}(E))$$

as the limit is filtered. As we know that the last term is quickly converging and that quickly converging towers commute with exact functors, it follows that

$$\begin{aligned} E \otimes \mathrm{Tot}\, CB^\bullet(E) &\simeq E \otimes \mathrm{Tot}(CB^{s2^\bullet}(E)) \simeq \mathrm{Tot}(CB^{s2^\bullet}(E) \otimes E) \\ &\simeq \lim_i \mathrm{Tot}_{\leq i}(CB^{s2^i}(E) \otimes E) \simeq \lim_i \mathrm{Tot}_{\leq i}(CB^i(E) \otimes E) \simeq \mathrm{Tot}(E \otimes CB^\bullet(E)) \simeq E, \end{aligned}$$

[4]see [1, Lect. 17, Lemma 5] for details on how to pass from checking on finite spectra to vanishing on all homologies.

where the last equivalence follows from Proposition 2.7. We conclude as in the proof of Proposition 3.1 1. that the natural map $L_E\mathbb{S} \to \operatorname{Tot} CB^\bullet(E))$ is an E-equivalence between E-local objects.
Now we can exploit our knowledge of the cofiber to show quick convergence. We claim that for any $i \geq 0$ the map $I^{\otimes 4s+i} \to I^{\otimes i}$ is null. Thus, the tower $\{\mathrm{cofib}(\operatorname{Tot} CB^\bullet(E) \to \operatorname{Tot}_{\leq n} CB^\bullet(E))\}_{n\geq 0}$ is $4s$-nilpotent and $\{\operatorname{Tot}_{\leq n} CB^\bullet(E)\}_{n\geq 0}$ is quickly converging as desired.
It suffices to check the claim for $i = 0$ since the above map is given by $I^{\otimes i} \otimes (I^{\otimes 4s} \to L_E\mathbb{S})$.
We can rewrite $I^{\otimes 4s} \to L_E\mathbb{S}$ as composition of the maps $I^{\otimes 4s} \to I^{\otimes 2s} \to I^{\otimes s} \to I \to L_E\mathbb{S}$. The first two maps are $Y_2 \to Y_1 \to Y_0$ and this is null since $\{Y_n\}$ is 2-nilpotent, showing the required. □

Remark 3.6. Details on how to exploit the horizontal vanishing line to get 2-nilpotence of the $\{Y_n\}_{n\geq 0}$ can be found in [1, Lect. 30]. In fact, the argument given there (attributed to Bousfield) and sketched here shows more generally that any cosimplicial object $X^\bullet$ whose associated spectral sequences have vanishing lines as in the above theorem is *proconstant*. More specifically, it is shown that $\{\operatorname{Tot}_{\leq 2^n s}(X^\bullet)\}_{n\geq 0}$ is quickly converging. However, this does not formally imply that the whole tower is too[5]. In general, to use this vanishing line criterion to get quick convergence, one needs further properties of the cofiber, as were given in the above situation. We thank Ishan Levy for explaining this last step to us.

All that is left to do now to prove the Smash Product Theorem is to show that the spectral sequence associated to $CB^\bullet(E_n)$ has the desired vanishing lines.

Proof of Thm. 3.2. Observe that $CB^\bullet(E_n)$ is quickly converging if and only if $CB^\bullet(E_n) \otimes \mathbb{S}_{(p)}$ is quickly converging because the towers are the same. As the class of spectra X such that $CB^\bullet(E_n) \otimes X$ is quickly converging is thick, it suffices, by the thick subcategory theorem, to show that there exists a type 0 spectrum X such that $CB^\bullet(E_n) \otimes X$ is quickly converging. This gives us the freedom to modify the tower we consider and the hope to replace the sphere by a spectrum for which the vanishing appears on the E_2-page already.
Recall that the E_2-page of the associated spectral sequence is given by the cohomology of the chain complex

$$E_{n*}(X) \to (E_n \otimes E_n)_*(X) \to (E_n \otimes E_n \otimes E_n)_*(X) \to \dots.$$

We want to show that its cohomology, as well as the cohomology of

$$E_{n*}(X \otimes F) \to (E_n \otimes E_n)_*(X \otimes F) \to (E_n \otimes E_n \otimes E_n)_*(X \otimes F) \to \dots$$

for any finite spectrum F, vanishes at some high enough degree.
Observe that this is a chain complex of $(E_{n*}, E_{n*}E_n)$-comodules and recall that these correspond to quasi-coherent sheaves on the moduli stack of formal groups of height less than or equal to n, $\mathcal{M}_{fg}^{\leq n}$. We denote the quasi-coherent sheaf associated to X by F_X. Under this correspondence, the above complex can be thought of as a free resolution of X by $(E_{n*}, E_{n*}E_n)$-comodules and its cohomology is given by $H^s(\mathcal{M}_{fg}^{\leq n}, F_{\Sigma^k X})$. We are thus reduced to finding a type 0 spectrum X such that $H^s(\mathcal{M}_{fg}^{\leq n}, F_{\Sigma^k X\otimes F})$ vanishes for all s big enough and all finite spectra F.
By filtering the stack $\mathcal{M}_{fg}^{\leq n}$ by height, one inductively reduces to showing that

$$H^s(\mathcal{M}_{fg}^k, i^*(F_X) \otimes G) = 0$$

for s big enough and all $G \in \mathrm{QCoh}(\mathcal{M}_{fg}^k)$, where $\mathcal{M}_{fg}^k$ denotes the moduli stack of formal groups of height exactly k and i^* is the restriction to the latter. For more details on this reduction see [1, Lect. 31].

[5] The cofiber will in general only be *protrivial* not nilpotent.

As field extensions are faithfully flat, it suffices to show vanishing of the above cohomology after base change to $\bar{\mathbb{F}}_p$. For this, consider the following pullback diagram

$$\begin{tikzcd} B\mathbb{G}_k \times \mathrm{Spec}(\bar{\mathbb{F}}_p) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{M}^k_{fg} \arrow[d] \\ \mathrm{Spec}(\bar{\mathbb{F}}_p.) \arrow[r, "i"] & \mathrm{Spec}(\mathbb{F}_p) \end{tikzcd}$$

Here $B\mathbb{G}_k$ is the classifying stack of the k^{th} Morava stabiliser group. The pullback is of this form as over an algebraically closed field all formal groups are isomorphic, so the moduli stack only records the isomorphisms.
Observe that $i^*(F_X)$ is in $\mathrm{QCoh}(B\mathbb{G}_k \times \mathrm{Spec}(\bar{\mathbb{F}}_p))$, i.e. it corresponds to the data of an $\bar{\mathbb{F}}_p$-vector space V together with a continuous $\mathbb{G}_k$-action and thus

$$H^*(\mathcal{M}^k_{fg}, F_X) \otimes_{\mathbb{F}_p} \bar{\mathbb{F}}_p = H^*(\mathbb{G}_k, V).$$

Summarising this discussion, we have reduced to finding a type 0 spectrum X and $s_0 > 0$ such that for all $1 \leq k \leq n$,

$$H^s(\mathbb{G}_k, V \otimes W) = 0 \text{ for all } s \geq s_0$$

where V denotes the $\bar{\mathbb{F}}_p$-vector space associated to X and W is any other $\bar{\mathbb{F}}_p$-vector space with a continuous group action (corresponding to the quasi-coherent sheaf G above).
Then, one can show that if $p > n + 1$, the group $\mathbb{G}_k$ has finite cohomological dimension and one can choose $X = \mathbb{S}_{(p)}$. In the other case, one has to work harder and dive into representation theory of profinite groups. The choice for X will turn out to be a modification of big enough complex projective spaces; see [3, Lem. 8.3.5-7, Sec. 8.4]. □

Remark 3.7 ([1, Lect. 31])**.** Let us give some more details on the above representation theoretic arguments. For intuition on why such a statement should hold, observe that $\mathbb{G}_k$ is a p-adic Lie group of rank k^2. In particular, it locally looks like $\mathbb{Z}_p^{\wedge k^2}$, and for any p-torsion module over $\mathbb{Z}_p^{\wedge k^2}$, the cohomology vanishes above degree k^2. However, this just shows why the cohomology vanishes for an open subgroup, it could very well happen (and it does) that it does not always vanish for the whole group at this rank.
The case distinction arises because, by an argument of Serre, if $\mathbb{G}_k$ is not of finite cohomological dimension, it must contain an element of order p. The elements of order p in $\mathbb{G}_k$ correspond to p^{th} roots of unity in $End(f)[p^{-1}]$ (f denotes the Honda formal group law). These exist only if $p - 1$ divides k. This gives the first case above.
When p is smaller, one can show that $\mathbb{G}_k$ contains at most a single conjugacy class V of order p subgroups (Skolem-Noether theorem). Moreover, for a $\mathbb{G}_k$-module M, the cohomology $H^*(\mathbb{G}_k, M)$ is bounded if V acts freely on M. Thus, one needs to find a spectrum X such that the associated $\mathbb{G}_k$-representation is free over V; see [3, Sec. 8.4].

4 Chromatic Convergence Theorem

We conclude by illustrating how quick convergence can be used to prove the Chromatic Convergence Theorem. Everything that follows was shown to me by Ishan Levy. Let us begin by recalling the statement.

Theorem 4.1 (Chromatic Convergence)**.** *Let X be a finite, p-local spectrum. Then the localisations $X \to L_n X$ assemble into an equivalence*

$$X \simeq \lim(\cdots \to L_2 X \to L_1 X \to L_0 X).$$

Remark 4.2. Exposition of this theorem and a different proof can be found in [1, Lect. 32]. There, it is shown that the limit of the tower of cofibers $\mathrm{cofib}(X \to L_n X)$ is trivial. This is done by showing that the map between the cofibers induces the zero map on MU-homology ([1, Lect. 32]) and thus increases the Adams-Novikov filtration. From this, one deduces the theorem by considering phantom maps below dimension n.

Proof. Again, the conclusion of the theorem is thick, thus it suffices to show the statement for $\mathbb{S}_{(p)}$. The proof will be based on the following two claims:

<u>Claim 1:</u> *Chromatic convergence holds for free $MU_{(p)}$-modules.*
Here a free $MU_{(p)}$-module is a sum of shifts of $MU_{(p)}$. The proof of this claim is an explicit calculation on homotopy groups and was blackboxed in this talk.

<u>Claim 2:</u> $CB^\bullet(L_n MU)$ *is quickly converging with limit* $L_n\mathbb{S}$.

Proof of Claim 2. As E_n is complex oriented, there is a ring map $L_n MU \to E_n$ and E_n is a $L_n MU$-module. Thus, $E_n \in \mathrm{Thick}^{\otimes}(L_n MU)$, as it is a retract of $E_n \otimes L_n MU$. In particular, $\mathrm{Thick}^{\otimes}(E_n) \subset \mathrm{Thick}^{\otimes}(L_n MU)$. As we have shown that $CB^\bullet(E_n)$ is quickly converging (Thm. 3.2), we know by Proposition 3.1 that $L_n\mathbb{S} \in \mathrm{Thick}^{\otimes}(E_n) \subset \mathrm{Thick}^{\otimes}(L_n MU)$. In other words, $L_n\mathbb{S}$ is $L_n MU$-nilpotent and, by Proposition 2.13, $CB^\bullet(L_n MU) = CB^\bullet(L_n MU)\otimes L_n\mathbb{S}$ is quickly converging towards $L_n\mathbb{S}$. ☐

Using these two claims, one proves chromatic convergence as follows:
Recall the Adams-Novikov filtration of $\mathbb{S}_{(p)}$, which arises from the cosimplicial diagram

$$MU_{(p)} \rightrightarrows MU_{(p)}^{\otimes 2} \rightrightarrows MU_{(p)}^{\otimes 3} \rightrightarrows \cdots$$

whose totalization is $\mathbb{S}_{(p)}$. Observe that $MU_{(p)}^{\otimes i}$ is a free $MU_{(p)}$-module in the above sense (i.e. it splits as a sum of shifts of $MU_{(p)}$) by the Thom isomorphism. In particular, by the first claim, each of the terms in the filtration satisfies chromatic convergence and $\lim_n L_n CB^\bullet(MU_{(p)}) \simeq CB^\bullet(MU_{(p)})$. Then, as desired, we can write the p-local sphere $\mathbb{S}_{(p)}$ as

$$\mathrm{Tot}(\lim_n L_n CB^\bullet(MU_{(p)})) \simeq \lim_n \mathrm{Tot}(L_n CB^\bullet(MU_{(p)})) \simeq \lim_n \mathrm{Tot}(CB^\bullet(L_n MU_{(p)})) \simeq \lim_n L_n\mathbb{S}.$$

The first equivalence combines the Adams-Novikov filtration and Claim 1, and then the two limits are interchanged. The third equivalence uses that L_n-localisation is smashing and the last equivalence is Claim 2. ☐

Remark 4.3. The proof of Claim 2 adapts to show that given a ring map $R \to S$, if S is descendable in some category then so is R.

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