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Overview on triangulated categories

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Abstract

This project introduces to a specific concept of category theory, namely that of triangulated categories such as they were defined by Verdier in his thesis. Based on existing literature, it aims to provide an overview of the basic properties of triangulated categories that is as complete as possible. Particular emphasis is given to illustrating the abstract notions with concrete examples and counter-examples. The main examples presented are the homotopy category and the derived category. Counter-examples include abelian categories that are not semisimple. Moreover, we discuss the question of how many distinct triangulations a triangulated category can admit and illustrate it by an explicit construction of a category with infinitely many different triangulations.

Contents

1 Introduction

In this project, we are interested in triangulated categories. Triangulated categories are additive categories with an additional structure, the triangulation. They provide the necessary abstract framework underlying many areas of modern mathematics such as algebraic geometry and homological algebra, in particular sheaf cohomology, but also representation theory (derived and stable module categories) or symplectic geometry (Fukaya categories). Moreover, via the derived categories, triangulated categories also have applications in less algebraic areas such as microlocal analysis or even in theoretical physics more specifically in string theory (D-branes and mirror symmetry).
Triangulated categories as we know them today were first introduced in 1963 in Jean-Louis Verdier's thesis [17]. Verdier was a PhD student of Alexander Grothendieck, who was working on his coherent duality theorems at this time. Grothendieck wanted to establish a suitable framework in which these theorems could be stated. In particular, a formal definition of the derived category was needed to refine the known theory of derived functors. He suggested to Verdier to construct this formalism in his thesis. Inspired by the derived category, Verdier discovered (invented?) the new and more general axiomatic properties of triangulated categories. Dieter Puppe had already introduced a similar notion in 1962, nowadays referred to as pretriangulated category. However, motivated by another example, that of stable homotopy categories, his definition was less complete (lacking axiom TR5). Although a summary of Verdier's thesis was published in 1963, the entire document wasn't available until 1996. For this reason, some of his results were rediscovered independently later and are generally attributed to the rediscoverers.
Still today, triangulated categories are present in modern mathematical research, which is not surprising given the numerous domains they have applications in. A few projects worth mentioning include Paul Balmer's work on tensor triangular geometry (see [3]), Amnon Neeman's research on K-theory of triangulated categories (see [14]) or research done at the university of Glasgow on the metrics and completions of triangulated categories (see [1]).

This project is structured into six main sections.
In the first section, entitled preliminaries, we will recall some basic results on category theory that will be needed later on. We remind the reader of the Yoneda lemma and a few of its consequences. Then, we focus on additive and abelian categories, recalling, in particular, the definition of kernel and cokernel and some properties of exact sequences.
In the second section, we define triangulated categories as Verdier introduced them in [17], give a first simple example, that of vector spaces, and investigate some basic properties following directly from the axiomatic. We are especially interested in the relationship between distinguished triangles and exact sequences and the question of when a distinguished triangle splits.
In chapter three, we introduce the homotopy category of an additive category with translation and differential objects. We follow the presentation proposed by Schapira and Kashiwara in [9]. This is the first "real" example of a triangulated category we will encounter. The homotopy category will quickly motivate the construction of the derived category, which, as was mentioned above, was the historic inspiration.
This is done in chapter four via the more general construction of the Verdier localisation. In this chapter, we define cohomology, give the construction of the Verdier localisation as Neeman presents it in [16] and explain why it is triangulated. Then, we describe how the derived category can be obtained as Verdier localisation of the homotopy category.
In the fifth chapter, we discuss examples of categories on which it is impossible to define a triangulation. In particular, we investigate the surprisingly restrictive relationship between abelian and triangulated categories, sketched as an exercise in Gelfand's and Manin's book [5], and analyse the example of the arrow category of a triangulated category given by Balmer [18].
Finally, in the last section, we wonder how many different triangulations a category can admit and give an example of a category with infinitely many distinct triangulations proposed by Balmer in [2].

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2 Preliminaries

We begin by recalling some results on category theory in general and include the proofs of results we will explicitly need in the rest of this paper. First, we introduce some definitions and notations and make a few reminders linked to the Yoneda lemma. Then, we define additive and abelian categories and evoke some properties of exact sequences. We follow chapters 1 and 8 of [9], in which more detailed explanations to the preliminaries can also be found.

2.1 Reminder of Basic Definitions & Results

Let us begin with the most fundamental definition, that of category.

Definition 2.1 (category)**.** A category $\mathcal{C}$ consists of:

- a class $Ob(\mathcal{C})$ called the objects of $\mathcal{C}$
- for any $X, Y \in Ob(\mathcal{C})$ a class $Hom_{\mathcal{C}}(X, Y)$ of morphisms from X to Y. We will often denote a morphism $f \in Hom_{\mathcal{C}}(X, Y)$ by $f : X \longrightarrow Y$.
- for any $X, Y, Z \in Ob(\mathcal{C})$ a map called the composition and denoted by:

$$\begin{aligned} Hom_{\mathcal{C}}(X, Y) \times Hom_{\mathcal{C}}(Y, Z) &\longrightarrow Hom_{\mathcal{C}}(X, Z) \\ (f, g) &\mapsto g \circ f \end{aligned}$$

such that $\circ$ is associative. This means that for any $f \in Hom_{\mathcal{C}}(X, Y)$, $g \in Hom_{\mathcal{C}}(Y, Z)$, $h \in Hom_{\mathcal{C}}(Z, W)$, it holds that $(h \circ g) \circ f = h \circ (g \circ f)$.
Moreover, for every $X \in Ob(\mathcal{C})$ there exists a morphism $id_X \in Hom_{\mathcal{C}}(X, X)$, called the identity of X, satisfying $id_X \circ f = f$ for all $f \in Hom_{\mathcal{C}}(Y, X)$ and $g \circ id_X = g$ for all $g \in Hom_{\mathcal{C}}(X, Y)$. Note that the identity of X is unique.

We define the opposite category:

Definition 2.2 (opposite category)**.** For a given category $\mathcal{C}$, the opposite category $\mathcal{C}^{op}$ is the category defined by $Ob(\mathcal{C}^{op}) = Ob(\mathcal{C})$ and $Hom_{\mathcal{C}^{op}}(X, Y) = Hom_{\mathcal{C}}(Y, X)$. The composition in the opposite category is denoted $\circ^{op}$ and defined for $f \in Hom_{\mathcal{C}^{op}}(X, Y)$ and $g \in Hom_{\mathcal{C}^{op}}(Y, Z)$ by $g \circ^{op} f = f \circ g$.

Recall the following standard definitions:

Definition 2.3 (isomorphism, monomorphism, epimorphism)**.** A morphism $f : X \longrightarrow Y$ is:

- an isomorphism if there exists $g : Y \longrightarrow X$ such that $g \circ f = id_X$ and $f \circ g = id_Y$. Such a g is unique and called an inverse of f.
- a monomorphism if for any pair of parallel morphisms $g_1, g_2 : Z \rightrightarrows X$, $f \circ g_1 = f \circ g_2$ implies $g_1 = g_2$. That is, for any object Z the map $f\circ : Hom(Z, X) \longrightarrow Hom(Z, Y)$ defined by $\varphi \mapsto f \circ \varphi$ is injective.
- an epimorphism if $f^{op} : Y^{op} \longrightarrow X^{op}$ is a monomorphism. Equivalently, f is an epimorphism, if for any pair of parallel morphisms $g_1, g_2 : Y \rightrightarrows Z$, $g_1 \circ f = g_2 \circ f$ implies $g_1 = g_2$. That is, for any object Z the map $\circ f : Hom(Y, Z) \longrightarrow Hom(X, Z)$ defined by $\varphi \mapsto \varphi \circ f$ is injective.

Definition 2.4 (initial/terminal/zero object)**.** An object $P \in \mathcal{C}$ is called initial if for any $X \in \mathcal{C}$, $Hom_{\mathcal{C}}(P, X) \cong \{\star\}$, in other words, for any object $X \in \mathcal{C}$, there is a unique morphism from P to X.
P is terminal if P is initial in $\mathcal{C}^{op}$. This means that for any $X \in \mathcal{C}$, $Hom_{\mathcal{C}}(X, P) \cong \{\star\}$.
P is a zero object if it is both initial and terminal. Such an object is often denoted by 0. For any $X, Y \in \mathcal{C}$, the morphism given by the composition $X \to 0 \to Y$ is called the zero morphism and denoted by $0 : X \to Y$.

Proposition 2.5. *Two initial (respectively terminal) objects are isomorphic.*

Proof. Let A, B be two initial objects. By definition there exist unique morphisms $f : A \longrightarrow B$, $g : B \longrightarrow A$ and $id_A : A \longrightarrow A$, $id_B : B \longrightarrow B$ are the unique morphisms from A to A respectively B to B. Notice that $f \circ g$ is also a morphism from B to B and $g \circ f$ from A to A. Thus, $f \circ g = id_B$ and $g \circ f = id_A$ i.e. A and B are isomorphic. The proof for terminal objects is analogous. $\square$

We remind the reader of a useful tool to compare categories:

Definition 2.6 (functor)**.** Consider $\mathcal{C}$ and $\mathcal{C}'$ two categories. A functor $F : \mathcal{C} \longrightarrow \mathcal{C}'$ consists of a map $F : Ob(\mathcal{C}) \longrightarrow Ob(\mathcal{C}')$ and of maps $F : Hom_{\mathcal{C}}(X, Y) \longrightarrow Hom_{\mathcal{C}'}(F(X), F(Y))$ for all $X, Y \in \mathcal{C}$, such that $F(id_X) = id_{F(X)}$ for all $X \in \mathcal{C}$ and $F(g \circ f) = F(g) \circ F(f)$ for all $f : X \to Y$, $g : Y \to Z$ in $\mathcal{C}$. A contravariant functor from $\mathcal{C}$ to $\mathcal{C}'$ is a functor from $\mathcal{C}^{op}$ to $\mathcal{C}'$. In other words, it satisfies $F(g \circ f) = F(f) \circ F(g)$.
For categories $\mathcal{C}$, $\mathcal{C}'$, $\mathcal{C}''$ and functors $F : \mathcal{C} \longrightarrow \mathcal{C}'$, $G : \mathcal{C}' \longrightarrow \mathcal{C}''$ the composition $G \circ F : \mathcal{C} \longrightarrow \mathcal{C}''$ is the functor defined by $(G \circ F)(X) = G(F(X))$ for all $X \in \mathcal{C}$ and $(G \circ F)(f) = G(F(f))$ for all morphisms f in $\mathcal{C}$.
The identity functor $id_{\mathcal{C}} : \mathcal{C} \longrightarrow \mathcal{C}$ is the identity on the objects and on the morphisms.
A functor $F : \mathcal{C} \longrightarrow \mathcal{C}'$ is invertible, or an isomorphism, if there exists a functor $G : \mathcal{C}' \longrightarrow \mathcal{C}$ such that $G \circ F = id_{\mathcal{C}}$ and $F \circ G = id_{\mathcal{C}'}$. In this case, we say that the categories $\mathcal{C}$ and $\mathcal{C}'$ are isomorphic. An isomorphism $F : \mathcal{C} \longrightarrow \mathcal{C}$ is called automorphism.

Example 2.7. For any category $\mathcal{C}$ and any object $X \in \mathcal{C}$, we define the functor $Hom_{\mathcal{C}}(X, -) : \mathcal{C} \longrightarrow \mathbf{Set}$, where **Set** denotes the category of sets. On the objects, it is given by $Y \mapsto Hom_{\mathcal{C}}(X, Y)$. A morphism $f : Y \longrightarrow Z$ is sent to

$$\begin{aligned} Hom_{\mathcal{C}}(X, f) : Hom_{\mathcal{C}}(X, Y) &\longrightarrow Hom_{\mathcal{C}}(X, Z) \\ \varphi &\mapsto f \circ \varphi \end{aligned}$$

When the category $\mathcal{C}$ is clear from the context, we write $Hom(X, -)$. We denote $Hom_{\mathcal{C}}(X, f)$ by f_*.
Dually, we define the contravariant functor $Hom_{\mathcal{C}}(-, X) : \mathcal{C}^{op} \longrightarrow \mathbf{Set}$. On the objects, it is given by $Y \mapsto Hom_{\mathcal{C}}(Y, X)$. A morphism $f : Y \longrightarrow Z$ is sent to

$$\begin{aligned} Hom_{\mathcal{C}}(f, X) : Hom_{\mathcal{C}}(Y, X) &\longrightarrow Hom_{\mathcal{C}}(Z, X) \\ \varphi &\mapsto \varphi \circ f \end{aligned}$$

We denote $Hom_{\mathcal{C}}(f, X)$ by f^*.

Proposition 2.8. *An automorphism $F : \mathcal{C} \longrightarrow \mathcal{C}$ maps initial objects to initial objects, terminal objects to terminal objects. In particular, if a zero object exists, $F(0) \cong 0$.*

Proof. Let A be an initial object, X any object. Note that $Hom_{\mathcal{C}}(F(A), X) \cong Hom_{\mathcal{C}}(A, F^{-1}(X)) \cong \{\star\}$ as A is initial. This shows that $F(A)$ is initial as well. Similarly, we show that for B a terminal object, $F(B)$ is terminal as well because $Hom_{\mathcal{C}}(X, F(B)) \cong Hom_{\mathcal{C}}(F^{-1}(X), B) \cong \{\star\}$ as B is terminal. Thus, since any two initial/terminal objects are isomorphic by proposition 2.5, $F(0) \cong 0$. $\square$

We define morphisms between functors:

Definition 2.9 (natural transformation)**.** Consider two categories $\mathcal{C}$ and $\mathcal{C}'$. Let F_1 and F_2 be two functors from $\mathcal{C}$ to $\mathcal{C}'$. A natural transformation $\eta : F_1 \longrightarrow F_2$ between F_1 and F_2 consists of a morphism $\eta_X : F_1(X) \longrightarrow F_2(X)$ for all $X \in \mathcal{C}$ such that for any morphism $f : X \to Y$ the diagram below commutes:

$$\begin{tikzcd} F_1(X) \arrow[r, "\eta_X"] \arrow[d, "F_1(f)"'] & F_2(X) \arrow[d, "F_2(f)"] \\ F_1(Y) \arrow[r, "\eta_Y"'] & F_2(Y) \end{tikzcd}$$

If $\eta : F_1 \longrightarrow F_2$ and $\theta : F_2 \longrightarrow F_3$ are natural transformations, we define the composition $\theta \circ \eta : F_1 \to F_3$ by $(\theta \circ \eta :)_X = \theta_X \circ \eta_X$.
We denote the set of natural transformations between two functors F and G by $Nat(F, G)$.

With these definitions, we can define the category of functors between two given categories $\mathcal{C}$ and $\mathcal{C}'$ which we denote $Fun(\mathcal{C}, \mathcal{C}')$. Its objects are the functors from $\mathcal{C}$ to $\mathcal{C}'$ and its morphisms are given by the natural transformations.

We will need the following functor:

Definition 2.10 ($h_{\mathcal{C}}$)**.** For any category $\mathcal{C}$, we define the functor $h_{\mathcal{C}} : \mathcal{C} \longrightarrow Fun(\mathcal{C}^{op}, \mathbf{Set})$. On the objects, it is given by $h_C(X) = Hom_C(-, X)$ and for a morphisms $f : X \longrightarrow Y$, $h_C(f)$ is the natural transformation consisting of the morphisms $h_C(f)_W = Hom(W, f)$.

We are now ready to recall a few important results.

Proposition 2.11 (Yoneda lemma)**.** *Let $F : \mathcal{C}^{op} \longrightarrow \mathbf{Set}$ be a functor and consider some $X \in \mathcal{C}$. Then, there is a bijection between the set of natural transformations from $Hom_{\mathcal{C}}(-, X)$ to F and the set $F(X)$, that is $Nat(Hom_{\mathcal{C}}(-, X), F) \cong F(X)$.*

Proof. Define the map $\phi : Nat(Hom_{\mathcal{C}}(-, X), F) \longrightarrow F(X)$ by $\phi(\eta) = \eta_X(id_X)$. We wish to define a map $\psi : F(X) \longrightarrow Nat(Hom_{\mathcal{C}}(-, X), F)$ inverse to ϕ. Construct ψ in the following way: To $s \in F(X)$ we associate a natural transformation $\psi(s) : Hom_{\mathcal{C}}(-, X) \longrightarrow F$ such that $\psi(s)_Y : Hom_{\mathcal{C}}(Y, X) \longrightarrow F(Y)$ is defined by $\psi(s)_Y(f) = F(f)(s)$.
For any $s \in F(X)$, it holds:

$$(\phi \circ \psi)(s) = \phi(\psi(s)) = \psi(s)_X(id_X) = F(id_X)(s) = id_{F(X)}(s) = s$$

and for any $\eta \in Nat(Hom_{\mathcal{C}}(-, X), F)$, it holds:

$$(\psi \circ \phi)(\eta) = \psi(\eta_X(id_X))$$

which is the natural transformation characterised by $\psi(\eta_X(id_X))_Y(f) = F(f)(\eta_X(id_X))$ for $f : Y \to X$. By naturality of η, for any $f : Y \to X$, the following square commutes:

$$\begin{tikzcd}
Hom(X,X) \arrow[r, "\eta_X"] \arrow[d, "Hom(f,X)"'] & F(X) \arrow[d, "F(f)"] \\
Hom(Y,X) \arrow[r, "\eta_Y"'] & F(Y)
\end{tikzcd}$$

$\eta_Y \circ Hom(f, X)(id_X) = \eta_Y(f) = F(f) \circ \eta_X(id_X) = \psi(\eta_X(id_X))_Y(f) \implies \psi(\eta_X(id_X))_Y = \eta_Y$.
Thus, ϕ and ψ are inverse to each other and they give the desired bijection. □

Definition 2.12 (fully faithful)**.** A functor $F : \mathcal{C} \longrightarrow \mathcal{C}'$ is fully faithful if we have a bijection

$$\begin{aligned} Hom_{\mathcal{C}}(X, Y) &\longrightarrow Hom_{\mathcal{C}'}(F(X), F(Y)) \\ f &\mapsto F(f) \end{aligned}$$

Corollary 2.13. *The functor $h_C : \mathcal{C} \longrightarrow Fun(\mathcal{C}^{op}, \mathbf{Set})$ defined in definition 2.10 is fully faithful.*

Proof. By the Yoneda lemma (prop. 2.11), for any $X, Y \in \mathcal{C}$ we have a bijection

$$\psi : h_C(Y)(X) = Hom_C(X, Y) \to Nat(Hom_{\mathcal{C}}(-, X), h_C(Y)) = Hom_{Fun(\mathcal{C}^{op}, \mathbf{Set})}(h_C(X), h_C(Y)),$$

where ψ is defined as in the proof of the Yoneda lemma. Notice that it is the map taking $f \in Hom_C(X, Y)$ to $h_C(f)$ as claimed. Indeed, $\psi(f) : Hom(-, X) \to Hom(-, Y)$ is the natural transformation defined by $\psi(f)_Z : Hom(Z, X) \to Hom(Z, Y)$ where $\psi(f)_Z(g) = Hom(g, Y)(f) = f \circ g = h_C(f)_Z(g)$. □

Corollary 2.14. *A morphism $f : X \longrightarrow Y$ is an isomorphism if and only if $Hom(W, f)$ is an isomorphism for all objects $W \in \mathcal{C}$.*

Proof. The "if" statement is clear by functoriality of $Hom(W, -)$. For the converse, notice that the hypothesis implies that $h_C(f)$ is an isomorphism. Since by corollary 2.13 h_C is fully faithful, f is an isomorphism. Indeed, let τ be the inverse of $h_C(f)$. Let $\phi : Hom(h_C(X), h_C(Y)) \longrightarrow Hom(X, Y)$ be the isomorphism inverse to $f \mapsto h_C(f)$. Then $\phi(\tau)$ is a left and right inverse of f since

$$h_C(f \circ \phi(\tau)) = h_C(f) \circ h_c(\phi(\tau)) = id_{h_C(Y)} = h_C(id_Y)$$

Applying ϕ yields $f \circ \phi(\tau) = \phi \circ h_C((f \circ \phi(\tau)) = \phi \circ h_C(id_Y) = id_Y$. Symmetrically, we find $f \circ \phi(\tau) = id_X$. Thus, f is an isomorphism with inverse $\phi(\tau)$. □

2.2 Additive & Abelian categories

In this subsection, we recall the definitions and basic properties of additive and abelian categories. Since, as we will see, triangulated categories are additive and the homotopy and derived categories are often constructed from abelian categories, it is important to have at least a basic understanding of these notions. Let us begin with additive categories. Before we can define additive categories, we need to introduce products and coproducts.

Definition 2.15 ((co)product)**.** Let $\mathcal{C}$ be a category and $\{X_i\}_{i\in I}$ any set of objects in $\mathcal{C}$. The product of $\{X_i\}_{i\in I}$, if it exists, is an object of $\mathcal{C}$, denoted $\Pi_{i\in I}X_i$, together with a set of morphisms $p_i : \Pi_{i\in I}X_i \to X_i$, called the projections. The product satisfies the following universal property: for any object $Y \in \mathcal{C}$ together with morphisms $f_i : Y \longrightarrow X_i$, there exists a unique morphism $\alpha : Y \longrightarrow \Pi_{i\in I}X_i$ such that for all $i \in I$ the following diagram commutes:

$$\begin{tikzcd} & Y \arrow[dl, "f_i"'] \arrow[d, "\exists!\alpha"] \\ X_i & \Pi_{i\in I}X_i \arrow[l, "p_i"] \end{tikzcd}$$

When I consists only of two elements, the product is denoted $X_1 \times X_2$ and the universal property rewrites as

$$\begin{tikzcd} & Y \arrow[dl, "f_1"'] \arrow[d, "\exists!\alpha"] \arrow[dr, "f_2"] & \\ X_1 & X_1 \times X_2 \arrow[l, "p_1"] \arrow[r, "p_2"'] & X_2 \end{tikzcd}$$

The coproduct denoted $\bigsqcup_{i\in I} X_i$ is the product in the opposite category $\mathcal{C}^{op}$. It is the object together with a set of morphisms (the inclusions) $\iota_i : X_i \to \bigsqcup_{i\in I} X_i$ such that for any other object $Y \in \mathcal{C}$ together with morphisms $f_i : X_i \longrightarrow Y$, there exists a unique morphism $\beta : \bigsqcup_{i\in I} X_i \longrightarrow Y$ such that for all i the following diagram commutes:

$$\begin{tikzcd} & Y \\ X_i \arrow[ur, "f_i"] \arrow[r, "\iota_i"'] & \bigsqcup_{i\in I} X_i \arrow[u, "\exists!\beta"'] \end{tikzcd}$$

Remark 2.16. It follows from the universal property that products and coproducts are unique up to unique isomorphism.

Definition 2.17 (additive category)**.** A category $\mathcal{C}$ is called additive if it satisfies the following:

1. for any $X, Y \in \mathcal{C}$, $Hom_{\mathcal{C}}(X,Y) \in Mod(\mathbb{Z})$ i.e. one can define an addition on morphisms with same domain and codomain which endows the set $Hom_{\mathcal{C}}(X,Y)$ with an abelian group structure.
2. the composition law $\circ$ is bilinear.
3. there exists a zero object, denoted by 0, in $\mathcal{C}$.
4. the category $\mathcal{C}$ admits finite coproducts.
5. the category $\mathcal{C}$ admits finite products.

Example 2.18. It is easy to check that the category $Mod(R)$ of modules over any ring R is additive or that **Ban**, the category of $\mathbb{C}-$Banach spaces together with linear continuous maps is additive. However, the category of rings with unit is not additive. Indeed, in this category, the initial object is $\mathbb{Z}$ and the terminal object is the ring 0. These rings are not isomorphic, so there cannot be a zero object.

The next proposition gives another characterisation of the finite products and coproducts in an additive category.

Proposition 2.19. *Let $\mathcal{C}$ be a category satisfying conditions 1)-3) in definition 2.17. Then, for any two objects X and Y in $\mathcal{C}$, there exists $Z \in \mathcal{C}$ and morphisms $i_1 : X \to Z$, $i_2 : Y \to Z$, $p_1 : Z \to X$ and $p_2 : Z \to Y$ satisfying*

$$p_1 \circ i_1 = id_X \ ; \ p_1 \circ i_2 = 0 \tag{1}$$

$$p_2 \circ i_2 = id_Y \ ; \ p_2 \circ i_1 = 0 \tag{2}$$

$$i_1 \circ p_1 + i_2 \circ p_2 = id_Z \tag{3}$$

if and only if condition 4) in definition 2.17 is satisfied if and only if condition 5) in definition 2.17 is satisfied. Moreover, the objects $X \coprod Y$, $X \times Y$ and Z are naturally isomorphic.

Proof. Assume condition 4). Set $Z = X \coprod Y$. Choose i_1 and i_2 to be the inclusions coming from the definition of the coproduct. Define $p_1 : X \coprod Y \longrightarrow X$ to be the identity on X and the zero morphism on Y and $p_2 : X \coprod Y \longrightarrow Y$ to be the identity on Y and the zero morphism on X. Equations (1) and (2) are satisfied by construction. Equation (3) follows from the fact that the only morphism f satisfying $f \circ i_1 = f \circ i_2$ is $id_{X \coprod Y}$.
Now suppose the existence of Z and the maps p_1, p_2, i_1, i_2. We check that Z together with i_1, i_2 verifies the universal property of the coproduct of X and Y. In that aim, consider W any object in $\mathcal{C}$ and suppose there are maps $f : X \longrightarrow W$, $g : Y \longrightarrow W$. We want to find a unique map $h : Z \longrightarrow W$ such that $h \circ i_1 = f$ and $h \circ i_2 = g$. Setting $h = f \circ p_1 + g \circ p_2$ works, and it is clear that it is the only possible choice. Thus, $Z \cong X \coprod Y$.
The remaining equivalences follow from the same arguments in the opposite category. □

Remark 2.20. In view of proposition 2.19, it is enough to require conditions 1)-4) or 1)-3) and 5) in the definition of an additive category. Furthermore, this proposition shows that finite coproducts and finite products must be isomorphic in additive categories. We introduce therefore the notation $X \oplus Y$ to denote the object that is simultaneously the product and coproduct of X and Y. We call it the direct sum of X and Y. For morphisms $f : X \to X'$, $g : Y \to Y'$, we denote by $f \oplus g : X \oplus Y \to X' \oplus Y'$ the unique map rendering the two following diagrams commutative:

$$\begin{tikzcd}
 & & X \oplus Y \arrow[dl, "p_X"'] \arrow[dr, "p_Y"] \arrow[dd, "f\oplus g"] & & \\
 & X \arrow[dl, "f"'] & & Y \arrow[dr, "g"] & \\
X' & & X' \oplus Y' \arrow[ll, "p_{X'}"] \arrow[rr, "p_{Y'}"'] & & Y'
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & X' \oplus Y' & & \\
 & X' \arrow[ur, "\iota_{X'}"] & & Y' \arrow[ul, "\iota_{Y'}"'] & \\
X \arrow[ur, "f"] \arrow[rr, "\iota_X"'] & & X \oplus Y \arrow[uu, "f\oplus g"] & & Y \arrow[ul, "g"'] \arrow[ll, "\iota_{Y'}"]
\end{tikzcd}$$

In the following we define a notion of functor that is compatible with the additional structure of additive categories.

Definition 2.21 (additive functor)**.** Let $F : \mathcal{C} \to \mathcal{C}'$ be a functor between two additive categories. F is an additive functor if for any $X, Y \in \mathcal{C}$ the map $Hom_{\mathcal{C}}(X, Y) \to Hom_{\mathcal{C}'}(F(X), F(Y))$ is a group homomorphism.

Example 2.22. Let $\mathcal{C}$ be an additive category and $X \in \mathcal{C}$. Then, $Hom(-, X)$ is an additive functor. To prove this, we need to check that for any morphisms $f, g : Y \longrightarrow Z$, $(f+g)^\star = f^\star + g^\star$. This follows from bilinearity of $\circ$. Indeed, for any $\varphi \in Hom(Y, X)$, $(f+g)^\star(\varphi) = \varphi \circ (f+g) = \varphi \circ f + \varphi \circ g = f^\star(\varphi) + g^\star(\varphi)$ as wanted. One can prove similarly that $Hom(X, -)$ is additive.

Proposition 2.23. *An additive functor $F : \mathcal{C} \to \mathcal{C}'$ preserves finite products and coproducts, meaning that $F(X \oplus Y) \cong F(X) \oplus F(Y)$ for all $X, Y \in \mathcal{C}$.*

Proof. Consider the direct sum $X \oplus Y$, the inclusions $\iota_X : X \to X \oplus Y$, $\iota_Y : Y \to X \oplus Y$ and the projections $p_X : X \oplus Y \to X$, $p_Y : X \oplus Y \to Y$. By proposition 2.19, these maps satisfy equations (1) to (3) defined in that same proposition.
It is sufficient to show that the maps $F(\iota_X) : F(X) \to F(X \oplus Y)$, $F(\iota_Y) : F(Y) \to F(X \oplus Y)$, $F(p_X) : F(X \oplus Y) \to F(X)$, $F(p_Y) : F(X \oplus Y) \to F(Y)$ satisfy equations (1) to (3) as well. Then, by the same proposition, $F(X \oplus Y) \cong F(X) \oplus F(Y)$.
Equations (1) and (2) are verified by functoriality of F and the fact that they are true for $\iota_X, \iota_Y, p_X, p_Y$. Since F is additive and by functoriality:

$$F(\iota_X) \circ F(p_X) + F(\iota_Y) \circ F(p_Y) = F(\iota_X \circ p_X + \iota_Y \circ p_Y) = F(id_{X \oplus Y}) = id_{F(X \oplus Y)},$$

where the last equality follows from (3) for $\iota_X, \iota_Y, p_X, p_Y$. As explained above, this concludes the proof. □

2.2.1 Kernel, Cokernel and Abelian Category

The kernel and cokernel are central objects in abelian categories. In order to define them, we first need to briefly look into what it means for a functor to be representable.

Definition 2.24 (representable functor)**.** Consider a category $\mathcal{C}$. A functor $F : C^{op} \longrightarrow \textbf{Set}$ is representable if there exists an object $X \in \mathcal{C}$ such that, for any $Y \in \mathcal{C}$, $F(Y) \cong Hom_{\mathcal{C}}(Y, X)$ functorially in Y, that is, if there is a natural isomorphism $Hom_{\mathcal{C}}(-, X) \cong F$. Then, X is called a representative of F.

The next proposition states that a representative X of a representable functor F is unique up to unique isomorphism.

Proposition 2.25. *The representative X of a representable functor F is determined uniquely up to unique isomorphism by the isomorphism $u : F \to Hom(-, X)$.*

Proof. Let Y be another representative of F. By definition, there is an isomorphism $u' : F \to Hom(-, Y)$. Define the natural transformation $\theta : Hom(-, X) \longrightarrow Hom(-, Y)$ by $\theta = u' \circ u^{-1}$. This yields an isomorphism between $Hom(-, X)$ and $Hom(-, Y)$. The natural transformation θ is uniquely determined by the isomorphisms u and u'. From corollary 2.13, we know that there exists an invertible map $\phi : Hom(Hom(-, X), Hom(-, Y)) \longrightarrow Hom(X, Y)$ sending isomorphisms to isomorphisms. Hence, $\phi(\theta)$ defines a unique isomorphism between X and Y. $\square$

Thus, it makes sense to speak of "the" representative of F and one can define objects as representatives of some functor. This is exactly the construction we will use to define the kernel and the cokernel.

Definition 2.26 ((co)kernel)**.** Suppose $\mathcal{C}$ is a category admitting a zero-object. For $f : X \longrightarrow Y$ a given morphism in $\mathcal{C}$, we define the kernel of f, denoted by $\ker(f)$, as a representative of the functor

$$\ker(Hom_{\mathcal{C}}(-, f)) : Z \mapsto \ker(Hom_{\mathcal{C}}(Z, X) \longrightarrow Hom_{\mathcal{C}}(Z, Y)).$$

The kernel on the right is defined in the way we know from group theory as

$$\ker(Hom_{\mathcal{C}}(Z, X) \to Hom_{\mathcal{C}}(Z, Y)) = \{\varphi \in Hom_{\mathcal{C}}(Z, X) \; |\varphi \circ f = 0\}.$$

Dually, we define the cokernel of f, denoted $\text{coker}(f)$, as representative of the functor

$$\ker(Hom_{\mathcal{C}}(f, -)) : Z \mapsto \ker(Hom_{\mathcal{C}}(Y, Z) \longrightarrow Hom_{\mathcal{C}}(X, Z)).$$

By proposition 2.25, the kernel and cokernel are unique up to unique isomorphism.

Remark 2.27. Equivalently, by definition of the representative, the kernel exists if and only if there is an object A in $\mathcal{C}$ and an isomorphism, functorial in Z, $Hom(Z, A) \cong \ker(Hom(Z, X) \longrightarrow Hom(Z, Y))$. Then, A is the kernel of f.
The isomorphism satisfied by the cokernel is $Hom(\text{coker}(f), Z) \cong \ker(Hom(X, Z) \longrightarrow Hom(Y, Z))$.

Remark 2.28. Since $\ker(Hom(Z, X) \longrightarrow Hom(Z, 0)) \cong Hom(Z, X)$, it follows from the isomorphisms that $\ker(X \longrightarrow 0) = X$ and $\ker(0 \to X) = 0$. Dually, $\text{coker}(0 \to X) = X$ and $\text{coker}(X \to 0) = 0$. Moreover, $\text{coker}(id_X) = \ker(id_X) = 0$.

Suppose $\ker(f)$ exists. Write F for the functor $\ker(Hom(-, f))$. Notice that, on the morphisms, F is defined by precomposition i.e. for $g : X' \longrightarrow Y'$, $F(g) : F(Y') \longrightarrow F(X')$ is defined by $F(g)(\varphi) = \varphi \circ g$.

By the Yoneda lemma, $Nat(Hom_{\mathcal{C}}(-, \ker(f)), F) \cong F(\ker(f))$. Denote the natural isomorphism given by representability of F, $Hom_{\mathcal{C}}(-, \ker(f)) \cong F$, by $\eta \in Nat(Hom_{\mathcal{C}}(-, \ker(f)), F)$. By the Yoneda lemma, η corresponds to an element $h \in F(\ker(f)) = \ker(Hom(\ker(f), X) \longrightarrow Hom(\ker(f), Y))$, i.e. η corresponds to a morphism $h : \ker(f) \to X$ satisfying $f \circ h = 0$.
Moreover, we claim that for any $Z \in \mathcal{C}$ and for any $t \in F(Z)$ i.e. any morphism $t : Z \to X$ such that $f \circ t = 0$, there exists a unique morphism $g : Z \to \ker(f)$ such that $h \circ g = t$.
Indeed, for any $Z \in \mathcal{C}$, we have an isomorphism $\eta_Z : Hom(Z, \ker(f)) \longrightarrow F(Z)$ given by $\eta_Z(g) = F(g)(h)$. To see this, observe that, keeping the notations from the proof of the Yoneda lemma, we have $\phi(\eta) = h$ and $\eta = \psi \circ \phi(\eta) = \psi(h)$. But $\psi(h)$ is precisely the natural transformation defined by $\phi(h)_Z(g) = F(g)(h)$. Denote the inverse morphism by $\eta_Z^{-1} : F(Z) \to Hom(Z, \ker(f))$. Set $g = \eta_Z^{-1}(t)$. We have that $t = \eta_Z(g) = F(g)(h) = h \circ g$ as wanted. This means that any morphism t satisfying $f \circ t = 0$ factors

uniquely through the kernel. This is visualised by the following diagram known as universal property of the kernel:

$$\begin{tikzcd}
\ker(f) \arrow[r, "h"] & X \arrow[r, bend left, "f"] \arrow[r, bend right, "0"'] & Y \\
& Z \arrow[u, "t"] \arrow[ul, dashed, "\exists! g"] \arrow[ur] &
\end{tikzcd}$$

We obtain dual statements for the cokernel. In the same way as we obtained h, we obtain a map $k : Y \longrightarrow$coker(f) and the universal property of the cokernel visualised by:

$$\begin{tikzcd}
\mathrm{coker}(f) \arrow[dr, dashed, "\exists! g"'] & Y \arrow[l, "k"'] \arrow[d, "t"] & X \arrow[l, bend right, "f"'] \arrow[l, "0"] \arrow[dl] \\
& A &
\end{tikzcd}$$

Proposition 2.29. *$h : ker(f) \longrightarrow X$ is a monomorphism and $k : Y \longrightarrow coker(f)$ is an epimorphism.*

Proof. We only prove the statement for h, the proof for k is dual. Let $g_1, g_2 : Z \longrightarrow X$ be two morphisms such that $h \circ g_1 = h \circ g_2$. Since $f \circ h = 0$, $f \circ h \circ g_1 = f \circ h \circ g_2 = 0$. Thus, by the universal property of the kernel, we obtain that $h \circ g_1$ and $h \circ g_2$ factor uniquely through h i.e. there is a unique map g such that $h \circ g = h \circ g_1 = h \circ g_2$. Both choices g_1 and g_2 for g work. Since g is unique, this implies $g_1 = g_2$ and h is a monomorphism. $\square$

Definition 2.30 ((co)image)**.** Let $\mathcal{C}$ be a category which admits kernels and cokernel. Let $f : X \to Y$ be a morphism in $\mathcal{C}$. The coimage of f denoted coim(f) is defined as coim(f) =coker(h) where $h : \ker(f) \to X$ is the morphism coming with the universal property of the kernel. The image of f is im(f) =ker(k) where $k : Y \to$ coker(f) where k comes from the universal property of the cokernel.

Consider the diagram:

$$\begin{tikzcd}
\ker(f) \arrow[r, "h"] & X \arrow[r, "f"] \arrow[d, "s"] & Y \arrow[r, "k"] & \mathrm{coker}(f) \\
& \mathrm{coim}(f) \arrow[ur, dashed, "\tilde{f}"] \arrow[r, dotted, "u"] & \mathrm{im}(f)) \arrow[u] &
\end{tikzcd}$$

$\tilde{f}$ is given by the universal property of the cokernel as $f \circ h = 0$ and coim(f) =coker(h). We have that $k \circ f = k \circ \tilde{f} \circ s = 0$ and s is an epimorphism, so $k \circ \tilde{f} = 0$. Therefore, $\tilde{f}$ factors through ker(k) =im(f) via the morphism $u : \mathrm{coim}(f) \to \mathrm{im}(f)$.
Let us illustrate on the category of abelian groups that these definitions coincide with the definitions one might know from group theory.

Example 2.31. Consider Ab the category of abelian groups and a morphism $\varphi : G_1 \to G_2$ of Ab. In group theory, the kernel of φ was defined by $\ker(\varphi) = \{g \in G_1 \mid \varphi(g) = 0\}$, the image was defined by $\mathrm{im}(\varphi) = \{g' \in G_2 \mid \exists\ g \in G_1\ s.t.\ \varphi(g) = g'\}$, the cokernel by $\mathrm{coker}(\varphi) = G_2/\mathrm{im}(\varphi)$ and the coimage by $\mathrm{coim}(\varphi) = G_1/\ker(\varphi)$.
Define h to be the inclusion of the kernel into G_1. Then, ker(φ) as just defined satisfies the universal property of the kernel. Indeed, let G_3 be another abelian group and consider a morphism $\psi : G_3 \longrightarrow G_1$ such that $\varphi \circ \psi = 0$. This condition implies that im(ψ) $\subseteq$ker(φ), so ψ factors uniquely through ker(φ) via the composition $G_3 \xrightarrow{\psi} \mathrm{im}(\psi) \xrightarrow{i} \ker(\varphi) \xrightarrow{h} G_1$ where i denotes the inclusion. Hence, the standard definition of kernel known from group theory coincides with the categorical definition given here. It remains to check that coker(φ) as defined here satisfies the universal property of the cokernel. We define $k : G_2 \longrightarrow \mathrm{coker}(\varphi)$ as the quotient map. We want to show that any morphism $\phi : G_2 \longrightarrow G_3$ such that $\phi \circ \varphi = 0$ factors uniquely through the cokernel. The condition $\phi \circ \varphi = 0$ implies that im(φ) $\subseteq$ker(ϕ). Notice that im(φ) =ker(k). Thus, the universal property of the quotient applies and assures that ϕ factors uniquely through the cokernel.
Since im(φ) =ker(k), the group theoretical and categorical definition of the image coincide too. Finally, $\mathrm{coim}(\varphi) = G_1/\ker(\varphi)$ which is precisely the cokernel of h because ker(φ) =im(h).

It is interesting to notice that in this example coim(φ) $\cong$im(φ). Indeed, by the first isomorphism theorem, coim(φ)=$G_1/\ker(\varphi)$ $\cong$im(φ). This motivates the definition of abelian category as given below.

Definition 2.32. Let $\mathcal{C}$ be an additive category. $\mathcal{C}$ is abelian if:

1. Any $f : X \to Y$ admits a kernel and a cokernel.
2. The morphism $u : \mathrm{coim}(f) \to \mathrm{im}(f)$ defined above is an isomorphism.

Example 2.33. As seen in example 2.31, abelian groups form an abelian category. More generally, for any ring R, $Mod(R)$ the category of R-modules is abelian.
The category of commutative rings with unit is not abelian, since as was explained in example 2.18, it is not even additive. We will later see more examples of categories that are additive but not abelian (c.f. thm. 6.3 and more generally section 6).

2.2.2 Exact sequences

An important notion in abelian categories is that of exact sequence. It is particularly important to us in this paper as we will see that the triangles of a triangulated category and exact sequences are closely related.

Definition 2.34 (exact sequence)**.** Let $\mathcal{A}$ be an abelian category. A sequence $X^k \xrightarrow{f_k} X^{k+1} \xrightarrow{f_{k+1}} \cdots \xrightarrow{f_{j-1}} X^j$ of objects X^i and morphisms f_i in $\mathcal{A}$ satisfying $f_{i+1} \circ f_i = 0$ $\forall i \in \{k, \cdots, j-1\} \subseteq \mathbb{Z}$ is called an exact sequence if $\mathrm{im}(f_i) \cong \ker(f_{i+1})$ $\forall i \in \{k, \cdots, j-1\}$.

An infinite sequence is exact if each one of its subsequences is exact.

An exact sequence of the form $0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$ is called short exact sequence.

Two sequences $X^k \xrightarrow{f_k} X^{k+1} \xrightarrow{f_{k+1}} \cdots \xrightarrow{f_{j-1}} X^j$ and $Y^k \xrightarrow{g_k} Y^{k+1} \xrightarrow{g_{k+1}} \cdots \xrightarrow{g_{j-1}} Y^j$ are said isomorphic, if there exists a family of isomorphisms $\{\psi_i\}_{k \leq i \leq j}$ such that for every $i \in \{k, \cdots, j-1\}$ $\psi_{i+1} \circ f_i = g_i \circ \psi_i$. An isomorphism of exact sequences is visualised by the following diagram:

$$\begin{tikzcd}
X^k \arrow[r, "f_k"] \arrow[d, "\psi_k"] & X^{k+1} \arrow[r, "f_{k+1}"] \arrow[d, "\psi_{k+1}"] & \cdots \arrow[r, "f_{j-1}"] & X^j \arrow[d, "\psi_j"] \\
Y^k \arrow[r, "g_k"] & Y^{k+1} \arrow[r, "g_{k+1}"] & \cdots \arrow[r, "g_{j-1}"] & Y^j
\end{tikzcd}$$

Remark 2.35. Note that for any morphism $f : X \to Y$ we get two short exact sequences:

$$0 \longrightarrow \ker(f) \longrightarrow X \longrightarrow \mathrm{coim}(f) \longrightarrow 0$$

and

$$0 \longrightarrow \mathrm{im}(f) \longrightarrow Y \longrightarrow \mathrm{coker}(f) \longrightarrow 0.$$

Let us state some elementary results on exact sequences, all of which we will need to discuss triangulated categories.

Proposition 2.36. *In an abelian category, for any morphism $f : X \to Y$, ker$(f) = 0$ if and only if f is a monomorphism. Dually, coker$(f) = 0$ if and only if f is an epimorphism.*

Proof. Suppose $\ker(f) = 0$. Let g_1, g_2 be maps such that $f \circ g_1 = f \circ g_2$. So $f \circ (g_1 - g_2) = 0$. By the universal property of the kernel, it follows that $g_1 - g_2$ factors through 0 i.e. $g_1 = g_2$ and f is a monomorphism. The following diagram sums up the situation:

$$\begin{tikzcd}
ker(f) = 0 \arrow[r, "h"] & X \arrow[r, bend left, "f"] \arrow[r, bend right, "0"'] & Y \\
& Z \arrow[ul, dashed, "\exists!\phi"] \arrow[u, "g_1 - g_2"] \arrow[ur] &
\end{tikzcd}$$

Conversely, suppose f is a monomorphism. Denote $h : ker(f) \longrightarrow X$ as in the diagram above. By definition $f \circ h = 0$. Since f is monic, this implies that $h = 0$. However, we know h is monic too, so for all maps g whose target is $\ker(f)$, $h \circ g = 0 \iff g = 0$. For this to be true, it must hold that $\ker(f) = 0$. Else, $g = id_{\ker(f)} \neq 0$ would statisfy $h \circ g = 0$ yielding a contradiction. The dual statement is obtained similarly (essentially by reversing arrows). $\square$

Corollary 2.37. *In a short exact sequence* $0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$ *, f is a monomorphism and g is an epimorphism.*

Proof. By definition, ker(f)=im(0) $= 0$ and im(g)=Z. For f, we conclude by proposition 2.36. Denote $k : Z \longrightarrow \mathrm{coker}(g)$, $\tilde{h} : \ker(k) \longrightarrow Z$ and remember that $\ker(k)$ =im(g)=Z and $k \circ \tilde{h} = 0$. Notice that, by definition, $\tilde{h} = id_Z$. Therefore, $k = 0$. k being an epimorphism now implies that $\mathrm{coker}(g) = 0$. Otherwise, $id_{\mathrm{coker}(g)} \circ k = 0$ but $id_{\mathrm{coker}(g)} \neq 0$ contradicting the fact that k is an epimorphism. □

Proposition 2.38. *In an abelian category, a morphism $f : X \to Y$ is an isomorphism if and only if it is both a monomorphism and an epimorphism. In particular, in view of proposition 2.36, f is an isomorphism if and only if ker(f) =coker(f)=0 if and only if the sequence* $0 \longrightarrow X \xrightarrow{f} Y \longrightarrow 0$ *is exact.*

Proof. It is evident from the definitions that an isomorphism is both an epimorphism and a monomorphism. For the converse, observe that by the universal property of the cokernel there is a unique map $X \to \mathrm{coker}(\ker(f) \to X) = \mathrm{coim}(f)$ and dually there is a unique map $\mathrm{im}(f) = \ker(Y \to \mathrm{coker}(f)) \to Y$. Since $\mathcal{A}$ is abelian, $\mathrm{coim}(f) \cong \mathrm{im}(f)$. Composing the three maps gives a unique factorisation of f as $X \longrightarrow \mathrm{im}(f) \longrightarrow Y$. We show that both arrows are isomorphisms. This will imply that f is an isomorphism too. Since f is an epimorphism, by proposition 2.36, coker(f)=0. Thus, $\ker(Y \to 0) = Y = \mathrm{im}(f)$. So the second arrow is an isomorphism. As f is also a monomorphism, ker(f)=0. Thus,
$\mathrm{coker}(\ker(f) \to X)$ =$\mathrm{coker}(0 \to X) = X = \mathrm{coim}(f) \cong$im($f$) which gives the second isomorphism. □

Remark 2.39. In general, this fails in non-abelian categories. For example, consider the category of commutative rings with unit. We claim that the inclusion $i : \mathbb{Z} \longrightarrow \mathbb{Q}$ is both an epimorphism and a monomorphism. However, it is evidently not an isomorphism. It is straightforward that the inclusion is a monomorphism. The fact that it is an epimorphism follows from the observation that any ring homomorphism $f : \mathbb{Q} \longrightarrow R$, where R is some ring, is uniquely determined by its values on $\mathbb{Z}$ since, for $a, b \in \mathbb{Z}$, $f(\frac{a}{b}) = f(a)f(\frac{1}{b}) = f(a)\frac{1}{f(b)}$.

Proposition 2.40. *Let* $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ *be a short exact sequence. The following conditions are equivalent:*

i) There exists a section $s : C \longrightarrow B$ such that $g \circ c = id_C$.

ii) There exists a retraction $r : B \longrightarrow A$ such that $r \circ f = id_A$.

iii) There exist a section $s : C \longrightarrow B$ and a retraction $r : B \longrightarrow A$ such that $id_B = f \circ r + s \circ g$.

iv) B is isomorphic to $A \oplus C$ and, more generally, $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ *is isomorphic to* $0 \longrightarrow A \xrightarrow{\iota_A} A \oplus C \xrightarrow{\pi_C} C \longrightarrow 0$ *where ι and π are the usual projections.*

Proof. It is clear that iv) implies the other ones since we can set $r = \pi_A \circ \varphi$ and $s = \varphi^{-1} \circ \iota_C$ where $\varphi : B \longrightarrow A \oplus C$ denotes the isomorphism.

iii) $\implies$ iv) is also straight forward:
Define $\varphi : B \longrightarrow A \oplus C$ by $\varphi = r + g$ and $\psi : A \oplus C \longrightarrow B$ by $\psi = f \circ \pi_A + s \circ \pi_C$.
Then, $\psi \circ \varphi = f \circ r + s \circ g = id_B$ by hypothesis and

$$\varphi \circ \psi = r \circ (f \circ \pi_A + s \circ \pi_C) + g \circ (f \circ \pi_A + s \circ \pi_C) = r \circ f \circ \pi_A + r \circ s \circ \pi_C + g \circ f \circ \pi_A + g \circ s \circ \pi_C = id_{A \oplus C}.$$

Here we used that in an additive category $\circ$ is bilinear, that by exactness $g \circ f = 0$, that r is a retraction and s a section and that $r \circ s = 0$. The last equality holds because of the following implications:

$$id_B - f \circ r = s \circ g \implies s - f \circ r \circ s = s \circ g \circ s = s \implies f \circ r \circ s = 0 \implies r \circ s = r \circ f \circ r \circ s = 0.$$

We show that i) $\implies$ iii).
Since $g = g \circ s \circ g$, $g \circ (id_B - s \circ g) = 0$. Therefore, $id_B - s \circ g$ factors through ker(g).

$$\begin{tikzcd}
\ker(g) \arrow[r, "h"] & B \arrow[r, shift left, "g"] \arrow[r, shift right, "0"'] & C \\
& B \arrow[u, "id_B - s\circ g"] \arrow[ul, dashed, "\exists!\varphi"] \arrow[ur] &
\end{tikzcd}$$

By exactness, $\ker(g) \cong \mathrm{im}(f)$. By definition, $\mathrm{im}(f)$=$\ker(k)$ where k is the map from B to $\mathrm{coker}(f)$. Recall that $k \circ f = 0$, thus we get:

$$\begin{tikzcd}
\mathrm{im}(f) \arrow[r, "h"] & B \arrow[r, shift left, "k"] \arrow[r, shift right, "0"'] & \mathrm{coker}(k) \\
& A \arrow[u, "f"] \arrow[ul, dashed, "\exists!\psi"] \arrow[ur] &
\end{tikzcd}$$

h denotes the same map as above as it comes with the universal property of $\ker(g) \cong \mathrm{im}(f)$ and is thus unique up to unique isomorphism.
Since we work in an abelian category, $\mathrm{im}(f) \cong \mathrm{coim}(f)$=$\mathrm{coker}(\tilde{h})$ where $\tilde{h}$ is the map from $\ker(f) = 0$ to A. By the universal property of the cokernel, we get:

$$\begin{tikzcd}
\mathrm{im}(f) \arrow[dr, dashed, "\exists!\phi"'] & A \arrow[l, "\tilde{k}"'] \arrow[d, "id_A"] & \ker(f) = 0 \arrow[l, shift right, "\tilde{h}"'] \arrow[l, shift left, "0"] \arrow[dl] \\
& A &
\end{tikzcd}$$

Observe that $h \circ \tilde{k} = f$ by definition of h and $\tilde{k}$. Thus $\psi = \tilde{k}$, $\phi \circ \psi = id_A$ and $h = f \circ \phi$. So:

$$\begin{tikzcd}
A \arrow[dr, "f"] & & \\
\mathrm{im}(f) \arrow[u, dashed, "\phi"] \arrow[r, "h"] & B \arrow[r, shift left, "g"] \arrow[r, shift right, "0"'] \arrow[d, "id_B - s\circ g"] & C \\
& B \arrow[ul, dashed, "\exists!\varphi"] \arrow[ur] &
\end{tikzcd}$$

$id_B - s \circ g = f \circ \phi \circ \varphi$, setting $r = \phi \circ \varphi$ we get $id_B = s \circ g + f \circ r$ as wanted.
Notice that $\varphi \circ f = \psi$. Since h is a monomorphism, it is sufficient to see that the compositions with h coincide: $h \circ \varphi \circ f = (id_B - s \circ g) \circ f = f = h \circ \psi$ because $g \circ f = 0$ by exactness. Therefore, $r \circ f = \phi \circ \varphi \circ f = \phi \circ \psi = id_A$. r is thus really a retraction, which concludes the proof.
The argument for *ii)* $\implies$ *iii)* is dual and will be omitted. All other implications are implied by the ones proven. $\square$

Definition 2.41 (split)**.** We say that a short exact sequence splits, if any of the previous equivalent conditions is met.

Remark 2.42. Notice that ϕ and ψ constructed in the proof give an isomorphism $A \cong \mathrm{im}(f)$. All that is left to check is $\psi \circ \phi = id_{\mathrm{im}(f)}$. But, $\psi \circ \phi \circ \psi = \psi \circ id_A = id_{\mathrm{im}(f)} \circ \psi$. Remembering that $\tilde{k} = \psi$ is an epimorphism, we deduce that $\psi \circ \phi = id_{\mathrm{im}(f)}$.
This reminds us of the isomorphism theorem we know from group or module theory: for $f : A \longrightarrow B$, $A/\ker(f) \cong \mathrm{im}(f)$. Here f is such that $\ker(f) = 0$, so we get an isomorphism $A \cong \mathrm{im}(f)$.

Motivated by this observation, we show that there is a more general statement resembling the first isomorphism theorem:

Proposition 2.43. *Consider an abelian category $\mathcal{A}$ and suppose there exists an invertible functor $T : \mathcal{A} \to \mathcal{A}$. Let $T^{-1}(Z) \xrightarrow{T^{-1}(h)} X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X) \xrightarrow{T(f)} T(Y)$ be an exact sequence. Then coker$(g) \cong$ker$(T(f))$ and ker$(h) \cong$coker(f).*

Proof. Observe that the cokernel of f can also be seen as push-out in the following diagram:

$$
\begin{tikzcd}
\mathrm{im}(f) \arrow[r, "\alpha"] \arrow[d] & Y \arrow[d, dashed, "\beta"] \\
0 \arrow[r, dashed] & \mathrm{coker}(f)
\end{tikzcd}
$$

where α is the inclusion of $\mathrm{im}(f)$ into Y as defined in the proof of proposition 2.38 and β is the morphism $Y \rightarrow$coker(f) given by the universal property of the cokernel.
We recall the universal property of the push-out visualised by the following diagram:

$$
\begin{tikzcd}
\mathrm{im}(f) \arrow[r, "\alpha"] \arrow[d] & Y \arrow[d, dashed, "\beta"] \arrow[ddr, bend left, "\phi"] & \\
0 \arrow[r, dashed] \arrow[drr, bend right] & \mathrm{coker}(f) \arrow[dr, dashed, "\exists ! \psi"] & \\
& & \forall A
\end{tikzcd}
\tag{4}
$$

In words, this means that for any object $A \in \mathcal{A}$ and any map $\phi : Y \to A$ such that $\phi \circ \alpha = 0$ there exists a unique morphism $\psi : \mathrm{coker}(f) \to Y$ such that $\phi = \psi \circ \beta$. Notice that $\phi \circ \alpha = 0$ if and only if $\phi \circ f = 0$. So we can reformulate the universal property of the push-out as: any morphism $\phi : Y \to A$ such that $\phi \circ f = 0$ factors uniquely through the cokernel of f. This is precisely the universal property of the cokernel and justifies why we can see the cokernel as push-out.
Remembering that $\mathrm{coim}(g) = \mathrm{coker}(\alpha : \mathrm{ker}(g) \longrightarrow Y)$ and that $\mathrm{im}(\alpha) = \mathrm{ker}(g)$, we see that $\mathrm{coim}(g)$ satisfies:

$$
\begin{tikzcd}
\mathrm{ker}(g) \arrow[r, "\alpha"] \arrow[d] & Y \arrow[d, dashed, "g"] \arrow[ddr, bend left, "\phi"] & \\
0 \arrow[r, dashed] \arrow[drr, bend right] & \mathrm{coim}(g) \arrow[dr, dashed, "\exists !"] & \\
& & \forall A
\end{tikzcd}
$$

By exactness of the sequence given in the proposition, $\mathrm{ker}(g) = \mathrm{im}(f)$ and since we work in an abelian category $\mathrm{coim}(g) \cong \mathrm{im}(g)$. Hence, the following square is a push-out as well.

$$
\begin{tikzcd}
\mathrm{im}(f) \arrow[r, "\alpha"] \arrow[d] & Y \arrow[d, dashed, "g"] \\
0 \arrow[r, dashed] & \mathrm{im}(g)
\end{tikzcd}
\tag{5}
$$

By exactness of the sequence given in the proposition, $g \circ \alpha = 0 = \beta \circ \alpha$. Hence, the universal property of the push-out applied to (4), where we set $A = \mathrm{im}(g)$ and $\phi = g$, yields the existence of a unique morphism $\psi : \mathrm{coker}(f) \longrightarrow \mathrm{im}(g)$ such that $g = \psi \circ \beta$. Applied to (5), where we set $A = \mathrm{coker}(f)$ and $\phi = \beta$, the universal property of the push-out yields the existence of a unique morphism $\varphi : \mathrm{im}(g) \longrightarrow \mathrm{coker}(f)$ such that $\varphi \circ g = \beta$. Now, $\varphi \circ \psi : \mathrm{coker}(f) \longrightarrow \mathrm{coker}(f)$ is a morphism satisfying $\varphi \circ \psi \circ \beta = \beta$. However, the identity $id_{\mathrm{coker}(f)}$ satisfies $id_{\mathrm{coker}(f)} \circ \beta = \beta$ as well. By the universal property of (4) with $A = \mathrm{coker}(f)$ and $\phi = \beta$, we know there is a unique map satisfying this relation and the identity works. Thus, $\varphi \circ \psi = id_{\mathrm{coker}(f)}$. By an analogous argument, it follows that $\psi \circ \varphi = id_{\mathrm{im}(g)}$. By exactness, $\mathrm{im}(g) = \mathrm{ker}(h)$ so $\mathrm{ker}(h) \cong \mathrm{coker}(f)$ as claimed. The situation is summed up by the following diagram:

$$
\begin{tikzcd}
\mathrm{im}(f) \arrow[r, "\alpha"] \arrow[d] & Y \arrow[d, "\beta"] \arrow[dr, bend left, "g"] \arrow[ddrr, bend left, "\beta"] & & \\
0 \arrow[r] \arrow[dr, bend right] \arrow[ddrrr, bend right] & \mathrm{coker}(f) \arrow[d, dashed, "\exists ! \psi"] \arrow[ddrr, equal] & & \\
& \mathrm{im}(g) \arrow[drr, dashed, "\exists ! \varphi"] & & \\
& & & \mathrm{coker} f
\end{tikzcd}
$$

where the upper square and the square in blue are the push-outs (4) and (5) respectively. The other isomorphism, $\mathrm{coker}(g) \cong \ker(T(f))$, is given by a symmetric argument. $\square$

Remark 2.44. Set $\mathcal{A} = Mod(R)$, the category of modules over some ring R. Then, in the setting of the proposition, $\mathrm{coker}(f)$ is defined as $Y/\mathrm{im}(f)$. Choose T to be the identity. Then, the first isomorphism theorem follows from this proposition. Indeed, by exactness and the proposition: ${}^{Z}\!/\!_{\ker(h)} \cong {}^{Z}\!/\!_{\mathrm{im}(g)} = \mathrm{coker}(g) \cong \ker(f) = \mathrm{im}(h)$.

The crucial point in this proof is that $\mathrm{coim}(g) \cong \mathrm{im}(g)$ in an abelian category. This is true by definition, so in some sense the first isomorphism theorem is intrinsic to abelian categories.

3 Introduction to Triangulated categories

We are now ready to define triangulated categories and discuss the axioms appearing in the definition. We follow [9] for the definitions. Most remarks on the axioms are inspired by [4]. Let us begin with a few preliminary definitions.

Definition 3.1 (category with translation)**.** A category with translation $(\mathcal{D}, T)$ is a category $\mathcal{D}$ endowed with an automorphism, that is an invertible functor $T : \mathcal{D} \to \mathcal{D}$. The functor T is called the translation functor or simply translation.

Definition 3.2 (triangle)**.** Let $(\mathcal{D}, T)$ be an additive category with translation. A triangle in $(\mathcal{D}, T)$ is a sequence of morphisms

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X).$$

A morphism of triangles is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "\alpha"] & Y \arrow[r, "v"] \arrow[d, "\beta"] & Z \arrow[r, "w"] \arrow[d, "\gamma"] & T(X) \arrow[d, "T(\alpha)"] \\ A \arrow[r, "a"] & B \arrow[r, "b"] & C \arrow[r, "c"] & T(A) \end{tikzcd}$$

Two triangles are isomorphic if each of the vertical arrows in the diagram above is invertible.

Remark 3.3. For $\epsilon_1, \epsilon_2, \epsilon_3 = \pm 1$, the triangle $X \xrightarrow{\epsilon_1 u} Y \xrightarrow{\epsilon_2 v} Z \xrightarrow{\epsilon_3 w} T(X)$ is isomorphic to $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$ if $\epsilon_1 \epsilon_2 \epsilon_3 = 1$, but this can fail when the product is -1.
Indeed, consider the following situation:

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "id"'] & Y \arrow[r, "v"] \arrow[d, "-id"'] & Z \arrow[r, "w"] \arrow[d, squiggly] & T(X) \arrow[d, "T(id)"] \\ X \arrow[r, "-u"'] & Y \arrow[r, "-v"'] & Z \arrow[r, "-w"'] & T(X) \end{tikzcd}$$

If we fix the first arrow to be id_X, the second one has to be $-id_Y$, in order for the first square to commute. The last arrow is necessarily $T(id_X) = id_{T(X)}$. If we want the second square to commute, the third arrow has to be id_Z, but then the third doesn't commute. So the above diagram can't be an isomorphism of triangles. The same problem arises if we start with $-id_X$.

Definition 3.4 (triangulated category)**.** A triangulated category is an additive category with translation denoted $(\mathcal{D}, T)$ endowed with a family of triangles, called distinguished triangles, satisfying the following axioms:

TR0 : Any triangle isomorphic to a distinguished triangle is distinguished.

TR1 : The triangle $X \Longrightarrow X \longrightarrow 0 \longrightarrow T(X)$ is distinguished. The first arrow denotes the identity.

TR2 : For any morphism $u : X \longrightarrow Y$, there exists a distinguished triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$ extending u.

TR3 : $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$ is distinguished $\Longleftrightarrow$ $Y \xrightarrow{-v} Z \xrightarrow{-w} T(X) \xrightarrow{-T(u)} T(Y)$ is distinguished.

TR4 : For any two distinguished triangles $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$ and $A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{c} T(A)$ such that there exists maps α and β making the first square of the following diagram commute, the dashed arrow exists.

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "\alpha"] & Y \arrow[r, "v"] \arrow[d, "\beta"] & Z \arrow[r, "w"] \arrow[d, dashed, "\exists\gamma"] & T(X) \arrow[d, "T(\alpha)"] \\ A \arrow[r, "a"] & B \arrow[r, "b"] & C \arrow[r, "c"] & T(A) \end{tikzcd}$$

Notice that γ is not necessarily unique.

TR5 : For any three distinguished triangles

$$X \xrightarrow{f} Y \xrightarrow{h} Z' \xrightarrow{m} T(X)$$

$$X \xrightarrow{g\circ f} Z \xrightarrow{l} Y' \xrightarrow{n} T(X)$$

$$Y \xrightarrow{g} Z \xrightarrow{k} X' \xrightarrow{p} T(Y)$$

there exists a distinguished triangle

$$Z' \xrightarrow{u} Y' \xrightarrow{v} X' \xrightarrow{w} T(Z')$$

such that the following diagram commutes:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "h"] \arrow[d, "g"] & Z' \arrow[r, "m"] \arrow[d, dashed, "u"] & T(X) \arrow[d, equal] \\
X \arrow[r, "g\circ f"] \arrow[d, "f"] & Z \arrow[r, "l"] \arrow[d, equal] & Y' \arrow[r, "n"] \arrow[d, dashed, "v"] & T(X) \arrow[d, "T(f)"] \\
Y \arrow[r, "g"] \arrow[d, "h"] & Z \arrow[r, "k"] \arrow[d, "l"] & X' \arrow[r, "p"] \arrow[d, equal] & T(Y) \arrow[d, "T(h)"] \\
Z' \arrow[r, dashed, "u"] & Y' \arrow[r, dashed, "v"] & X' \arrow[r, dashed, "w"] & T(Z')
\end{tikzcd}$$

In other words, for any three distinguished triangles extending the morphisms f, g and $g \circ f$, we want the existence of morphisms u, v such that (id_X, g, u) and (f, id_Z, v) are morphisms of triangles and $Z' \xrightarrow{u} Y' \xrightarrow{v} X' \xrightarrow{T(h)\circ p} T(Z')$ is distinguished. In this more concise but less visual way TR5 was stated in Verdier's thesis.

These are the (slightly rewritten) axioms Verdier introduced in his thesis. They are neither independent nor optimal. We will discuss some of the dependencies between the axioms in subsection 3.4. A category satisfying only axioms TR0-TR4 in known as pretriangulated category. It is still an open problem if axioms TR0-TR4 imply axiom TR5.
We will now try to get a better understanding of the meaning behind each of these axioms and the reasons why they are required in the definition. Throughout this project, we will see that the distinguished triangles resemble exact sequences in many ways. Viewing distinguished triangles as generalisation of exact sequences to non-abelian settings is a good approach to understand TR5 as the following remark will show.

Remark 3.5 (on TR5)**.** One can think of TR5 as a generalisation of the third isomorphism theorem for modules which states that for modules $X \subseteq Y \subseteq Z$,

$$\left(Z/X\right)/\left(Y/X\right) \cong Z/Y.$$

Indeed, seeing the distinguished triangles as short exact sequences, we get injections such that $X \subseteq Y \subseteq Z$ and isomorphisms $Z' \cong Y/X$, $Y' \cong Z/X$ and $X' \cong Z/Y$. TR5 gives us the existence of a distinguished triangle

$$Y/X \overset{u}{\dashrightarrow} Z/X \overset{v}{\dashrightarrow} Z/Y \overset{w}{\dashrightarrow} T(Y/X),$$

which, in its turn seen as an exact sequence, gives an isomorphism $Z/Y \cong \left(Z/X\right)/\left(Y/X\right)$.

Remark 3.6 (on TR3)**.** TR3 asserts that if any one adjacent triple of morphisms in the following infinite sequence is distinguished

$$\begin{tikzcd}
T^{-n}(Z) \arrow[r, "(-1)^nT^{-n}(w)"] & \cdots \arrow[r, "-T^{-1}(u)"] & T^{-1}(Y) \arrow[r, "-T^{-1}(v)"] & T^{-1}(Z) \arrow[dll, "-T^{-1}(w)"'] \\
& X \arrow[r, "u"] & Y \arrow[r, "v"] & Z \arrow[dlll, "w"] \\
T(X) \arrow[r, "-T(w)"'] & T(Y) \arrow[r] & \cdots \arrow[r, "(-1)^{n-1}T^{n-1}(w)"'] & T^n(X) \arrow[r, "(-1)^nT^n(u)"'] & \cdots
\end{tikzcd}$$

then all adjacent triples will be distinguished.

Remark 3.7 (on TR4)**.** The position of the missing arrow doesn't matter. More precisely TR4 also gives the following two diagrams:

$$\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[d, "\phi"'] & B \arrow[r, "\beta"] \arrow[d, dashed, "\exists\psi"'] & C \arrow[r, "\gamma"] \arrow[d, "\delta"'] & T(A) \arrow[d, "T(\phi)"] \\
X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z \arrow[r, "h"'] & T(X)
\end{tikzcd} \tag{6}$$

and

$$\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[d, dashed, "\exists\phi"'] & B \arrow[r, "\beta"] \arrow[d, "\psi"'] & C \arrow[r, "\gamma"] \arrow[d, "\delta"'] & T(A) \arrow[d, dashed, "\exists T(\phi)"] \\
X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z \arrow[r, "h"'] & T(X)
\end{tikzcd} \tag{7}$$

Indeed, by TR3, the triangle $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} T(A)$ is distinguished if and only if

the following triangle is a distinguished triangle $$T^{-1}(C) \xrightarrow{-T^{-1}(\gamma)} A \xrightarrow{-\alpha} B \xrightarrow{-\beta} C \tag{8}$$

if and only if the following triangle is distinguished $$B \xrightarrow{-\beta} C \xrightarrow{-\gamma} T(A) \xrightarrow{-T(\alpha)} T(B) \tag{9}$$

Similarly, the triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X)$ is a distinguished triangle if and only if

the following triangle is a distinguished triangle $$T^{-1}(Z) \xrightarrow{-T^{-1}(h)} X \xrightarrow{-f} Y \xrightarrow{-g} Z \tag{10}$$

if and only if the following triangle is distinguished $$Y \xrightarrow{-g} Z \xrightarrow{-h} T(X) \xrightarrow{-T(f)} T(Y) \tag{11}$$

Suppose ϕ and δ as in (6) exist and apply TR4 to (8) and (10). We get

$$\begin{tikzcd}
T^{-1}(C) \arrow[r, "-T^{-1}(\gamma)"] \arrow[d, "T^{-1}(\delta)"'] & A \arrow[r, "-\alpha"] \arrow[d, "\phi"'] & B \arrow[r, "-\beta"] \arrow[d, dashed, "\exists\psi"'] & C \arrow[d, "\delta"] \\
T^{-1}(Z) \arrow[r, "-T^{-1}(h)"'] & X \arrow[r, "-f"'] & Y \arrow[r, "-g"'] & Z
\end{tikzcd}$$

It is easy to see that ψ obtained here also completes (6).
Similarly, suppose ψ and δ exists as in (7) and apply TR4 to (9) and (11). We get

$$\begin{tikzcd}
B \arrow[r, "-\beta"] \arrow[d, "\psi"'] & C \arrow[r, "-\gamma"] \arrow[d, "\delta"'] & T(A) \arrow[r, "-T(\alpha)"] \arrow[d, dashed, "\exists\phi"'] & T(B) \arrow[d, "T(\psi)"] \\
Y \arrow[r, "-g"'] & Z \arrow[r, "-h"'] & T(X) \arrow[r, "-T(f)"'] & T(Y)
\end{tikzcd} \tag{12}$$

We see that $T^{-1}(\phi)$ completes (7). More specifically, from (12), it follows that $T(\psi) \circ -T(\alpha) = -T(f) \circ \phi$ i.e. $T(\psi \circ \alpha) = T(f) \circ \phi$ and $\phi \circ -\gamma = -h \circ \delta$ i.e. $\phi \circ \gamma = h \circ \delta$. Noticing $T(T^{-1}(\phi) = \phi$, the second equation is exactly what we want in (7) and applying T^{-1} to the first we get $\psi \circ \alpha = f \circ T^{-1}(\phi)$ making the other square of (7) commute.

Remark 3.8 (non-uniqueness in TR4)**.** We have already mentioned that the morphism given by TR4 is not necessarily unique. We will now give a concrete example. We will see in proposition 3.27 that the direct sum of distinguished triangles is distinguished. Moreover, by applying TR3 twice to $X = X \longrightarrow 0 \longrightarrow T(X)$, we see that $T^{-1}(X) \longrightarrow 0 \longrightarrow X = X$ is a distinguished triangle and in proposition 3.12 we will show that $0 \longrightarrow Y = Y \longrightarrow 0$ is also a distinguished triangle. Therefore, the direct sum $T^{-1}(X) \xrightarrow{0} Y \xrightarrow{i} X \oplus Y \xrightarrow{p} X$, where i and p are the inclusions and projection given by the universal property of $\oplus$, is distinguished as well. Consider

$$\begin{tikzcd}
T^{-1}(X) \arrow[r, "0"] \arrow[d, equal] & Y \arrow[r, "i"] \arrow[d, equal] & X \oplus Y \arrow[r, "p"] \arrow[d, dashed, "\alpha"] & X \arrow[d, equal] \\
T^{-1}(X) \arrow[r, "0"'] & Y \arrow[r, "i"'] & X \oplus Y \arrow[r, "p"] & X
\end{tikzcd}$$

then α can be any morphism of the form $\begin{pmatrix} id_X & 0 \\ f & id_Y \end{pmatrix}$ where f is any morphism from X to Y.

It is a natural question to wonder whether the opposite category of a triangulated category is triangulated. It turns out that this is the case. This is a useful property, as it enables us to obtain certain results by duality.

Definition 3.9. Let $(\mathcal{D}, T)$ be a triangulated category. We say a triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$ is anti-distinguished in $\mathcal{D}$ if the triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{-w} T(X)$ is distinguished.

One defines the opposite category $(\mathcal{D}^{op}, T^{op})$ as follows: $\mathcal{D}^{op}$ is the opposite category in the usual sense and $T^{op} = op \circ T^{-1} \circ op^{-1}$. Notice that for any $X \in \mathcal{D}$, $(T^{op})^n(X) = T^{-n}(X)$, in particular $T^{op^{-1}}(X) = T(X)$.

We define a triangle $T^{op^{-1}}(X) \xrightarrow{w^{op}} Z \xrightarrow{v^{op}} Y \xrightarrow{u^{op}} X$ to be distinguished in $(\mathcal{D}^{op}, T^{op})$ if and only if the triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$ is anti-distinguished in $(\mathcal{D}, T)$.

It is easy to check that this defines a triangulation on $\mathcal{D}^{op}$ and that anti-distinguished triangles form a triangulation of $\mathcal{D}$. Moreover, in many categories this new triangulation will differ from the initial one, that is to say, anti-distinguished triangles form a triangulation on $\mathcal{D}$ that doesn't coincide with that formed by distinguished triangles. We will look at these results in more generality in section 7.

3.1 A First Example & Exact Sequences

A first simple example of a triangulated category is the category $\mathbf{Vect}_k$ of vector spaces over a field k with the identity as translation functor. We define the distinguished triangles to be sequences of the form $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X$ such that $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X \xrightarrow{u} Y$ is exact. Recall $\mathbf{Vect}_k$ is abelian, so exactness is well-defined. This example illustrates that the axioms for triangulated categories are nicely compatible with exact sequences. After having acquired some more knowledge on triangulated categories, we will show in proposition 3.19 that, with the choice of the identity for T, the triangulation defined above is the only triangulation we can endow $\mathbf{Vect}_k$ with. This confirms the intuition that distinguished triangles generalise exact sequences to non-abelian settings. Let us now first see why this category is triangulated. Before proving this, recall a preliminary result on exact sequences:

Proposition 3.10. *In* $\mathbf{Vect}_k$*, any exact sequence of the form*

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X \xrightarrow{u} Y$$

is isomorphic to the exact sequence

$$X \xrightarrow{u} Y \xrightarrow{\iota_v} ker(u) \oplus im(v) \xrightarrow{\pi_u} X \xrightarrow{u} Y$$

where ι_v *is defined by* $\iota_v(y) = 0 \oplus v(y)$ *and* π_u *is the projection given by* $\pi_u(x \oplus v(y)) = x$.

Proof. First of all, we check that $X \xrightarrow{u} Y \xrightarrow{\iota_v} \ker(u) \oplus \mathrm{im}(v) \xrightarrow{\pi_u} X \xrightarrow{u} Y$ is an exact sequence: $\mathrm{im}(u) = \ker(\iota_v)$ because, by supposed exactness of the first sequence, $\mathrm{im}(u) = \ker(v) = \ker(\iota_v)$. Clearly, $\mathrm{im}(\iota_v) = \{0\} \oplus \mathrm{im}(v) = \ker(\pi_u)$ and $\mathrm{im}(\pi_u) = \ker(u)$ by construction.
We will now build an isomorphism between Z and $\ker(u)\oplus\mathrm{im}(v)$. We denote the elements of $\ker(u)\oplus\mathrm{im}(v)$ by $x \oplus v(y)$ keeping in mind that here x is supposed to be in the kernel of u.
By exactness of the first sequence, $\ker(u) = \mathrm{im}(w)$. Clearly w seen as map from Z to $\ker(u)$ is surjective. By linear algebra, we can write $Z = \ker(w) \oplus S$ where S is a subspace of Z, supplement of $\ker(w)$. We claim that the restiction of w to S, $w|_S : S \longrightarrow \ker(u)$, is a linear isomorphism. Injectivity follows from the observation that $\ker(w|_S)$=$\ker(w) \cap S = 0$. For surjectivity, we know that for any $x \in \ker(u)$, there exists $z \in Z$ such that $w(z) = x$ by surjectivity of w. Write $z = z_i + z_s$ with $z_i \in \ker(w)$ and $z_s \in S$. Then, $w(z) = w(z_i + z_s) = w(z_i) + w(z_s) = w|_S(z_s)$. Therefore, $w|_S$ is invertible. Denote its inverse by w^{-1} . The map w^{-1} is linear and satisfies $w \circ w^{-1} = id_{\ker(u)}$.
Define $\phi : \ker(u) \oplus \mathrm{im}(v) \longrightarrow Z$ by $\phi(x \oplus v(y)) = w^{-1}(x) + v(y)$. This map is clearly linear as u, v, w^{-1} are linear. The following diagram is now commutative:

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r, "v"] \arrow[d, equal] & Z \arrow[r, "w"] & X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[d, equal] \\
X \arrow[r, "u"] & Y \arrow[r, "\iota"] & \ker(u) \oplus \mathrm{im}(g) \arrow[r, "\pi"] \arrow[u, "\phi"] & X \arrow[r, "u"] & Y
\end{tikzcd}$$

Indeed, $\phi \circ \iota(y) = \phi(0 \oplus v(y)) = v(y)$ and $w \circ \phi(x \oplus v(y)) = w(w^{-1}(x) + v(y)) = x = \pi(x)$. The five-lemma applies and gives us that ϕ is an isomorphism as wanted.

□

Proposition 3.11. *The category* $(\mathbf{Vect}_k, Id)$ *is triangulated. Distinguished triangles are given by sequences of the form* $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X$ *such that* $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X \xrightarrow{u} Y$ *is exact.*

Proof. Although the proof is just checking axioms using linear algebra, it will be described in detail to get accustomed to the somewhat abstract axioms. We need to verify the axioms one by one.

TR0: It is clear that any sequence isomorphic to an exact sequence is exact since composing by isomorphisms corresponds to a change of basis. More precisely, if $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X \xrightarrow{u} Y$ is a sequence that is isomorphic to an exact sequence $X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} X' \xrightarrow{u'} Y'$ via the following diagram:

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "\psi_u"] & Y \arrow[r, "v"] \arrow[d, "\psi_v"] & Z \arrow[r, "w"] \arrow[d, "\psi_w"] & X \arrow[r, "u"] \arrow[d, "\psi_u"] & Y \arrow[d, "\psi_v"] \\
X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & X' \arrow[r, "u'"] & Y'
\end{tikzcd}$$

then $u = \psi_v^{-1} \circ u' \circ \psi_u$ and $v = \psi_w^{-1} \circ v' \circ \psi_v$. We know that $\ker(v')$=$\mathrm{im}(u')$. An easy calculation shows that this implies $\ker(v) = \ker(\psi_v^{-1} \circ v' \circ \psi_u) = \mathrm{im}(\psi_w^{-1} \circ u' \circ \psi_v) = \mathrm{im}(u)$. A similar reasoning for w shows that the upper row in the diagram is also exact.

TR1: $X = X \longrightarrow 0 \longrightarrow X = X$ is clearly an exact sequence of the desired form.

TR2: Given a map $f : X \longrightarrow Y$, we can extend it to the exact sequence

$$X \xrightarrow{f} Y \xrightarrow{\iota} \ker(f) \oplus \mathrm{coker}(f) \xrightarrow{\pi} X \xrightarrow{f} Y$$

where ι and π are the natural maps i.e. $\iota(y) = 0 \oplus [y]$ where $[y]$ denotes the equivalence class of y in $Y/\mathrm{im}(f)$ and $\pi(x \oplus [y]) = x$. This sequence is exact because $\ker(\iota) = \mathrm{im}(f)$, $\mathrm{im}(\iota) = 0\oplus\mathrm{coker}(f)$ =$\ker(\pi)$ and $\mathrm{im}(\pi)$ =$\ker(f)$ by construction.

TR3: TR3 is straightforward in this case, as T is simply the identity.

TR4: By proposition 3.10, keeping its notations, we can suppose without loss of generality that we have the following diagram:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "a"] & Y \arrow[r, "\iota_g"] \arrow[d, "b"] & \ker(f) \oplus \operatorname{im}(g) \arrow[r, "\pi_f"] \arrow[d, dashed, "\alpha"] & X \arrow[r, "f"] \arrow[d, "a"] & Y \arrow[d, "b"] \\
\tilde{X} \arrow[r, "u"] & \tilde{Y} \arrow[r, "\iota_v"] & \ker(u) \oplus \operatorname{im}(v) \arrow[r, "\pi_u"] & \tilde{X} \arrow[r, "u"] & \tilde{Y}
\end{tikzcd}$$

and we want to prove that the dashed arrow exists. Notice that $\iota_g : Y \longrightarrow 0 \oplus \operatorname{im}(g)$ is surjective. We can thus construct a right inverse of ι_g which we denote ι_g^{-1} using the same reasoning as for w^{-1} in the proof of 3.10.

Define α by $\alpha(x \oplus g(y)) = a \circ \pi_f(x \oplus 0) \oplus v \circ b \circ \iota_g^{-1}(0 \oplus g(y))$. We know that $a \circ \pi_f(x \oplus 0) \in \ker(u)$ as $u \circ a \circ \pi_f(x \oplus 0) = b \circ f \circ \pi_f(x \oplus 0) = 0$ so α is well-defined.

Let us check that with this definition of α the diagram is commutative. Observe that if y, y' are such that $\iota_g^{-1}(0 \oplus g(y)) = \iota_g^{-1}(0 \oplus g(y'))$ i.e. $y = y' + \tilde{y}$ for some $\tilde{y} \in \ker(g) = \operatorname{im}(f)$ so $\tilde{y} = f(x)$, then

$$\iota_v \circ b(y) = \iota_v \circ b(y' + f(x)) = \iota_v \circ b(y') + \iota_v \circ b(f(x)) = \iota_v \circ b(y') + \iota_v \circ u \circ a(x) = \iota_v \circ b(y')$$

by exactness of the second row and commutativity of the first square. In other words, if $\alpha \circ \iota_g(y) = \alpha \circ \iota_g(y')$, then $\iota_v \circ b(y) = \iota_v \circ b(y')$. Thus, it is sufficient to check commutativity for y such that $\iota_g^{-1}(0 \oplus g(y)) = y$, which is satisfied by construction. Notice that here α is unique (from the construction, it is clear that there is no other way to define it).

TR5: Again, we suppose without loss of generality that the sequences are like those in the proof of axiom TR4 above. Suppose we have maps $f : X \to Y$, $h : Y \to Y'$, $g : Y \to Z$, $l : Z \to Z'$ and $k : Z \to \tilde{Z}$ and three exact sequences:

$$X \xrightarrow{f} Y \xrightarrow{\iota_h} \ker(f) \oplus \operatorname{im}(h) \xrightarrow{\pi_f} X \tag{13}$$

$$Y \xrightarrow{g} Z \xrightarrow{\iota_k} \ker(g) \oplus \operatorname{im}(k) \xrightarrow{\pi_g} Y \tag{14}$$

$$X \xrightarrow{g \circ f} Z \xrightarrow{\iota_l} \ker(g \circ f) \oplus \operatorname{im}(l) \xrightarrow{\pi_{g \circ f}} X \tag{15}$$

We want to get an exact sequence

$$\ker(f) \oplus \operatorname{im}(h) \xrightarrow{u} \ker(g \circ f) \oplus \operatorname{im}(l) \xrightarrow{v} \ker(f) \oplus \operatorname{im}(g) \xrightarrow{w} \ker(g) \oplus \operatorname{im}(k)$$

such that

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "\iota_h"] \arrow[d, "g"] & \ker(f) \oplus \operatorname{im}(h) \arrow[r, "\pi_f"] \arrow[d, dashed, "u"] & X \arrow[d, equal] \\
X \arrow[r, "g \circ f"] \arrow[d, "f"] & Z \arrow[r, "\iota_l"] \arrow[d, equal] & \ker(g \circ f) \oplus \operatorname{im}(l) \arrow[r, "\pi_{g \circ f}"] \arrow[d, dashed, "v"] & X \arrow[d, "f"] \\
Y \arrow[r, "g"] \arrow[d, "\iota_h"] & Z \arrow[r, "\iota_k"] \arrow[d, "\iota_l"] & \ker(g) \oplus \operatorname{im}(k) \arrow[r, "\pi_g"] \arrow[d, equal] & Y \arrow[d, "\iota_h"] \\
\ker(f) \oplus \operatorname{im}(h) \arrow[r, dashed, "u"] & \ker(g \circ f) \oplus \operatorname{im}(l) \arrow[r, dashed, "v"] & \ker(g) \oplus \operatorname{im}(k) \arrow[r, dashed, "w"] & \ker(f) \oplus \operatorname{im}(h)
\end{tikzcd}$$

Let u be the map given by TR4 applied to the upper rectangle (defined like α in the proof of TR4) i.e. $u(x \oplus h(y)) = x \oplus l \circ g(h^{-1} \circ h(y))$ and v given by TR4 applied to the middle rectangle i.e. $v(x \oplus l(z)) = f(x) \oplus k(l^{-1} \circ l(z))$. For w we have no other choice than setting $w = \iota_h \circ \pi_g$. By construction, these choices make the diagram commute. We need to check the bottom line is exact.

We claim that $\ker(v) = \ker(f) \oplus \operatorname{im}(l \circ g) = \operatorname{im}(u)$.

Indeed, $v(x \oplus l(z)) = 0$ if and only if $f(x) = 0$ (i.e. $x \in \ker(f)$) and $k(l^{-1} \circ l(z)) = 0$. The latter holds if and only if $l^{-1} \circ l(z)$ is in the kernel of k. By definition of ι_k, $\ker(\iota_k)$ $=\ker(k)$ so by exactness of (14), $\ker(k)$ $=\operatorname{im}(g)$. If $l^{-1} \circ l(z)$ is in the image of g, i.e. there exists some $y \in Y$ such that $g(y) = l^{-1} \circ l(z)$, then $l(z) = l(g(y))$ because $l(g(y)) = l \circ l^{-1} \circ l(z) = id_{\tilde{Z}} \circ l(z)$. So $k(l^{-1} \circ l(z)) = 0$ if and only if $l(z) \in \operatorname{im}(l \circ g)$. It is straightforward that $\operatorname{im}(u) \subseteq \ker(f) \oplus \operatorname{im}(l \circ g)$. The reverse inclusion holds because for any element $x \oplus l \circ g(y) \in \ker(f) \oplus \operatorname{im}(l \circ g)$,$x \oplus h(y)$ is an element satisfying $u(x \oplus h(y)) = x \oplus l \circ g(h^{-1} \circ h(y)) = x \oplus l \circ g(y)$. This is evident for those $y \in Y$ such that $h^{-1} \circ h(y) = y$. For the others, there exists y' such that $h^{-1} \circ h(y') = y' = h^{-1} \circ h(y)$. This implies $y = y' + \tilde{y}$ with $\tilde{y}$ $\in\ker(h)$. By exactness of (13), $\ker(h)$ $=\ker(\iota_h)$ $=\operatorname{im}(f)$, so there exists some $x \in X$ such that $\tilde{y} = f(x)$. Hence, $y = y' + f(x)$ and

$$l \circ g(y) = l \circ g(y' + f(x)) = l \circ g(y') + l \circ g(f(x)) = l \circ g(y') = l \circ g(h^{-1} \circ h(y))$$

as $l \circ g \circ f = 0$ by exactness of (15). Therefore, $u(x \oplus h(y)) = x \oplus l \circ g(y)$.

We check that $\ker(w) = \ker(\iota_h \circ \pi_g) = (\operatorname{im}(f) \cap \ker(\mathrm{g})) \oplus \operatorname{im}(k)$.

We know that $w(y \oplus k(z)) = 0 \oplus h(y) = 0$ if and only if y is in the kernel of h which by exactness is the image of f. Then, by definition of w such a y is both in $\operatorname{im}(f)$ and in $\ker(g)$. This yields the desired equality. Notice that also $(\operatorname{im}(f) \cap \ker(g)) \oplus \operatorname{im}(k) = \operatorname{im}(v)$. Indeed, by an analogous argument as was used to prove $\operatorname{im}(u) = \ker(f) \oplus \operatorname{im}(l \circ g)$, $v(0 \oplus \operatorname{im}(l) = 0 \oplus \operatorname{im}(k)$. Moreover, for x $\in\ker(g \circ f)$, $v(x \oplus 0) = f(x) \oplus 0$ is in the kernel of g and the image of f and any $y \in \operatorname{im}(f) \cap \ker(\mathrm{g})$ must be of the form $f(x)$ with x $\in\ker(g \circ f)$. Therefore, $\ker(w) = (\operatorname{im}(f) \cap \ker(\mathrm{g})) \oplus \operatorname{im}(k) = \operatorname{im}(v)$.

Finally, observe that $\operatorname{im}(w) = 0 \oplus \operatorname{im}(h|_{\ker(g)})$. This follows directly from the definition of w as $w(y \oplus k(z)) = 0 \oplus h(y)$ where y $\in\ker(g)$.

Recall that in the proof of $\operatorname{im}(u) = \ker(f) \oplus \operatorname{im}(l \circ g)$, it was shown that $l \circ g(h^{-1} \circ h(y)) = l \circ g(y)$. Therefore, $\operatorname{im}(w) \subseteq \ker(u) = 0 \oplus \ker(l \circ g \circ h^{-1})$ and $\ker(l \circ g) = \ker(l \circ g \circ h^{-1} \circ h)$. For the reverse inclusion, notice that by exactness of (15), $\ker(l) = \operatorname{im}(g \circ f)$. This implies that $l \circ g(y) = 0$ if and only if $g(y)$ is in the image of $g \circ f$ i.e. $g(y) = g \circ f(x)$ for some $x \in X$. This can hold only if $y = f(x) + y'$ where $y' \in \ker(g)$. Thus, $\ker(l \circ g)$ decomposes as the sum $\ker(g) + \operatorname{im}(f)$ and $\ker(u) = 0 \oplus \operatorname{im}(h|_{\ker(g)+\operatorname{im}(f)})$.But notice that $\operatorname{im}(h|_{\ker(g)+\operatorname{im}(f)}) = \operatorname{im}(h|_{\ker(g)})$ since $\ker(h) = \operatorname{im}(f)$ by exactness of (13). Thus $\operatorname{im}(w) = \ker(u)$.

This proves that the bottom row is an exact sequence, so a distinguished triangle and thus concludes the proof of TR5.

□

Having this example in mind, we shall now draw some first general consequences for arbitrary triangulated categories. Most of the following results and their proofs can be found in [9]. Until the end of the section, we fix a triangulated category $(\mathcal{D}, T)$.

Proposition 3.12. *Let Z be any object of $\mathcal{D}$, then the sequence* $0 \xrightarrow{0} Z \xrightarrow{id_Z} Z \xrightarrow{0} 0$ *is a distinguished triangle.*

Proof. By proposition 2.8, $T(0) \cong 0$ and notice that a triangle with $T(0)$ is always isomorphic to a triangle where we replace $T(0)$ by 0. By TR3, $0 \xrightarrow{0} Z \xrightarrow{id_Z} Z \xrightarrow{0} 0$ is distinguished if and only if $Z \xrightarrow{-id_Z} Z \longrightarrow T(0) \longrightarrow T(Z)$ is distinguished. Multiplying the first line by -1 and by the above remark on $T(0)$, this triangle is isomorphic to $Z \xrightarrow{id_Z} Z \longrightarrow 0 \longrightarrow T(Z)$, which is distinguished by TR1. □

The following propositions are interesting as they make a further link to exact sequences.

Proposition 3.13. *For any distinguished triangle* $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$ *the composition $v \circ u = 0$.*

Proof. We have the following commutative diagram:

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r] \arrow[d, "u"] & 0 \arrow[r] \arrow[d, dashed, "\gamma"] & T(X) \arrow[d, equal] \\
X \arrow[r, "u"] & Y \arrow[r, "v"] & Z \arrow[r, "w"] & T(X)
\end{tikzcd}$$

where the upper row is a distinguished triangle by TR1 and γ exists by TR4. Hence, $v \circ u = \gamma \circ 0 = 0$. $\square$

Remark 3.14. Using TR3, it follows that also $w \circ v = 0$ and $T(u) \circ w = 0$. More generally, the composition of any two adjacent morphisms in the long sequence defined in remark 3.6 is 0.

Notice that the composition of two adjacent morphisms being zero is a necessary condition for a sequence in an abelian category to be exact. In this sense, one could say that distinguished triangles are similar to exact sequences. One can wonder if this link can be made more explicit. Are there functors between triangulated and abelian categories that send distinguished triangles to exact sequences? This question will now be discussed.

Definition 3.15. A functor $F : \mathcal{D} \longrightarrow \mathcal{C}$ from a triangulated category $\mathcal{D}$ to an abelian category $\mathcal{C}$ is called cohomological if for any distinguished triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$ the sequence $F(X) \xrightarrow{F(u)} F(Y) \xrightarrow{F(v)} F(Z)$ is exact in the sense that $\ker(F(v)) = \mathrm{im}(F(u))$.

Proposition 3.16. *For any $K \in \mathcal{D}$, the functor $Hom_{\mathcal{D}}(K, -)$ is cohomological.*

Proof. Consider a distinguished triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$ and an object $K \in \mathcal{D}$. Write u_* for the morphism given by $Hom_{\mathcal{D}}(K, u)$ and v_* for $Hom_{\mathcal{D}}(K, v)$. We want to show that

$$Hom_{\mathcal{D}}(K, X) \xrightarrow{u_*} Hom_{\mathcal{D}}(K, Y) \xrightarrow{v_*} Hom_{\mathcal{D}}(K, Z)$$

is exact. From proposition 3.13, $\mathrm{im}(u_*) \subseteq \ker(v_*)$. For the reverse inclusion, we need that for any $\phi \in \ker(v_*)$ i.e. such that $v \circ \phi = 0$, there exists $\psi \in Hom_{\mathcal{D}}(K, X)$ such that $\phi = u \circ \psi$. In other words, we want the existence of the dashed arrows in the diagram below:

$$\begin{tikzcd}
K \arrow[r, equal] \arrow[d, dashed] & K \arrow[r] \arrow[d, "\phi"] & 0 \arrow[r] \arrow[d] & T(K) \arrow[d, dashed] \\
X \arrow[r, "u"] & Y \arrow[r, "v"] & Z \arrow[r, "w"] & T(X)
\end{tikzcd}$$

This follows by TR1 and remark 3.7 where it was shown that the position of the missing arrow in axiom TR4 doesn't matter. $\square$

Remark 3.17 (long exact sequence)**.** Applying proposition 3.16 to the infinite sequence given in remark 3.6, we get a long exact sequence

$$\cdots \longrightarrow Hom(K, T^{-1}(Z)) \longrightarrow Hom(K, X) \longrightarrow Hom(K, Y) \longrightarrow Hom(K, Z) \longrightarrow Hom(K, T(X)) \longrightarrow \cdots$$

Moreover, there is a dual statement for $Hom_{\mathcal{D}}(-, K)$.

Proposition 3.18. *For any object $K \in \mathcal{D}$, the functor $Hom_{\mathcal{D}}(-, K)$ is cohomological. In other words, for any distinguished triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(x)$, the sequence*

$$Hom_{\mathcal{D}}(X, K) \xleftarrow{u^*} Hom_{\mathcal{D}}(Y, K) \xleftarrow{v^*} Hom_{\mathcal{D}}(Z, K)$$

is exact. Here, u^* denotes $Hom_{\mathcal{D}}(u, K)$ and v^* denotes $Hom_{\mathcal{D}}(v, K)$.

Proof. We need $\ker(u^*) = \mathrm{im}(v^*)$. We proceed by double inclusion.
$\underline{\mathrm{im}(v^*) \subseteq \ker(u^*)}$: As the triangle is distinguished, proposition 3.13 implies that $v \circ u = 0$. If $\varphi \in \mathrm{im}(v^*)$, i.e. $\exists \psi$ s.t. $\psi \circ v = \varphi$, then $\varphi \in \ker(u^*)$ as $\varphi \circ u = \psi \circ v \circ u = 0$.
$\underline{\ker(\circ u) \subseteq \mathrm{im}(\circ v)}$: For any $\varphi \in \ker(u^*)$, i.e. $\varphi \in Hom_{\mathcal{D}}(Y, K)$ such that $\varphi \circ u = 0$, we need the existence of a morphism $\psi \in Hom_{\mathcal{D}}(Z, K)$ such that $\psi \circ v = \varphi$. In other words, we want the following diagram to make sense:

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "0"'] & Y \arrow[r, "v"] \arrow[d, "\varphi"'] & Z \arrow[r, "w"] \arrow[d, dashed, "\exists\psi"] & T(X) \arrow[d, "0"] \\
0 \arrow[r] & K \arrow[r, "id_K"'] & K \arrow[r] & 0
\end{tikzcd}$$

Since the two rows are distinguished triangles (by assumption and by proposition 3.12), the existence of ψ is given by TR4 and $Hom_{\mathcal{D}}(-, K)$ is exact. $\square$

These properties are very useful as they allow us to transfer certain results on distinguished triangles to results on exact sequences of $\mathbb{Z}$-modules (recall $\mathcal{D}$ is additive) of which we often have a better understanding. We now have all the tools at hand to prove what we had claimed earlier:

Proposition 3.19. *The set of triangles described in proposition 3.11 is the only possible choice to make* $(\mathbf{Vect}_k, Id)$ *a triangulated category. In particular, in this category any distinguished triangle is necessarily an exact sequence.*

Proof. Consider an arbitrary triangulation $\mathcal{T}'$ on $(\mathbf{Vect}_k, Id)$ and let $X \xrightarrow{\ u\ } Y \xrightarrow{\ v\ } Z \xrightarrow{\ w\ } X$ be a triangle in $\mathcal{T}'$ (i.e. it is distinguished with respect to $\mathcal{T}'$). We will show that this triangle has to be an exact sequence and hence $\mathcal{T}'$ is included in the triangulation defined in 3.11, which we will denote by $\mathcal{T}$.
By proposition 3.13, $v \circ u = 0 = w \circ v$. Since by TR3 $Y \xrightarrow{\ -v\ } Z \xrightarrow{\ -w\ } X \xrightarrow{\ -u\ } Y$ is distinguished as well, by the same proposition $u \circ w = 0$. Thus, $\mathrm{im}(u) \subseteq \ker(v)$, $\mathrm{im}(v) \subseteq \ker(w)$ and $\mathrm{im}(w) \subseteq \ker(u)$.
To obtain the reverse inclusions, remember that by proposition 3.18 the sequence

$$Hom(X, K) \xleftarrow{\ \circ u\ } Hom(Y, K) \xleftarrow{\ \circ v\ } Hom(Z, K) \tag{16}$$

is exact for any $K \in (\mathbf{Vect}_k, Id)$.
Choose $K = \mathrm{coker}(u) = Y/\mathrm{im}(u)$. Let $p : Y \twoheadrightarrow K$ be the quotient map. Notice that $p \circ u = 0$ since $\ker(p) = \mathrm{im}(u)$ i.e. $p \in \ker(u^*)$. By exactness of (16), $p \in \mathrm{im}(v^*)$ i.e. there exists $\phi \in Hom(Z, K)$ such that $\phi \circ v = p$. We deduce that $\ker(v) \subseteq \ker(p) = \mathrm{im}(u)$. Since, by TR3, $Y \xrightarrow{-v} Z \xrightarrow{-w} X \xrightarrow{-u} Y$ and $Z \xrightarrow{w} X \xrightarrow{u} Y \xrightarrow{v} Z$ are also distinguished, we obtain the remaining inclusions analogously.
To prove equality of $\mathcal{T}'$ and $\mathcal{T}$, suppose for the sake of contradiction that the inclusion is strict. By proposition 3.10, $\mathcal{T}$ is the family of all exact sequences of the form $X \xrightarrow{u} Y \xrightarrow{\iota_v} \ker(u) \oplus \mathrm{im}(v) \xrightarrow{\pi_u} X \xrightarrow{u} Y$ and sequences isomorphic to those. As $\mathcal{T}'$ is a triangulation, it is closed under isomorphism and since it is contained in $\mathcal{T}$ it must be the family of some sequences of the form $X \xrightarrow{u} Y \xrightarrow{\iota_v} \ker(u) \oplus \mathrm{im}(v) \xrightarrow{\pi_u} X \xrightarrow{u} Y$ and sequences isomorphic to those. Since the inclusion is supposed strict, there must be a sequence $X' \xrightarrow{f} Y' \xrightarrow{\iota_g} \ker(f) \oplus \mathrm{im}(g) \xrightarrow{\pi_f} X \xrightarrow{f} Y$ that is not in $\mathcal{T}'$. However, repeating the proof of 3.10, we deduce that any distinguished triangle extending f must be isomorphic to this sequence. Thus, by TR0, there cannot be any triangle extending f in $\mathcal{T}'$, but this contradicts TR2. $\square$

3.2 Isomorphisms & Distinguished Triangles

In this subsection, we investigate the relations between isomorphisms and distinguished triangles. We prove a statement resembling the five-lemma for distinguished triangles, the proof given here is a slightly modified version of that in [9], and give an equivalent condition for a morphism to be an isomorphism in terms of triangles. The latter is inspired by [16].

Definition 3.20. We say a triangle (not necessarily distinguished) is exact if it induces a long exact sequence when applying $Hom(K, -)$ and $Hom(-, K)$ for every object $K \in \mathcal{D}$.

Remark 3.21. By proposition 3.16 and proposition 3.18, all distinguished triangles are exact. We notice that anti-distinguished triangles are also exact, but not necessarily distinguished (see [13]).

Proposition 3.22. *Consider the following commutative diagram:*

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "\alpha"'] & Y \arrow[r, "v"] \arrow[d, "\beta"'] & Z \arrow[r, "w"] \arrow[d, "\gamma"'] & T(X) \arrow[d, "T(\alpha)"'] \\
A \arrow[r, "a"'] & B \arrow[r, "b"'] & C \arrow[r, "c"'] & T(A)
\end{tikzcd}$$

Suppose that both lines are exact triangles and that α and β are isomorphisms. Then, γ is an isomorphism too. In particular, in view of remark 3.21, if both lines of the diagram are distinguished triangles, given the isomorphisms α and β, γ exists by TR4 and is an isomorphism as well.

Proof. By assumption, we have the following diagram with exact lines:

$$\begin{tikzcd}
Hom(K,X) \arrow[r, "u_*"] \arrow[d, "\alpha_*"'] & Hom(K,Y) \arrow[r, "v_*"] \arrow[d, "\beta_*"'] & Hom(K,Z) \arrow[r, "w_*"] \arrow[d, "\gamma_*"'] & Hom(K,T(X)) \arrow[r, "T(u)_*"] \arrow[d, "T(\alpha)_*"'] & Hom(K,T(Y)) \arrow[d, "T(\beta)_*"'] \\
Hom(K,A) \arrow[r, "a_*"'] & Hom(K,B) \arrow[r, "b_*"'] & Hom(K,C) \arrow[r, "c_*"'] & Hom(K,T(A)) \arrow[r, "T(a)_*"'] & Hom(K,T(B))
\end{tikzcd}$$

Notice that α_* and β_* are isomorphisms, since functors take isomorphisms to isomorphisms. For the same reasons, $T(\alpha)_*$ and $T(\beta)_*$ are isomorphisms. By the five lemma, γ_* is an isomorphism. Since this holds for all objects K, we conclude, by corollary 2.14, that γ is an isomorphism.

$\square$

Remark 3.23. By a similar argument as in remark 3.7, if both lines are distinguished, we also have that if α and γ are ismorphisms, β will be an isomorphism as well, or if β and γ are isomorphisms then so will be α.

Corollary 3.24. *For any morphism $f : X \longrightarrow Y$ the triangle given by TR2 extending f is unique up to isomorphism of triangles.*

Proof. Suppose $X \xrightarrow{f} Y \longrightarrow Z \longrightarrow T(X)$ and $X \xrightarrow{f} Y \longrightarrow Z' \longrightarrow T(X)$ are two distinguished triangles extending f. By proposition 3.22, the dashed arrow in the following diagram exists and is an isomorphism:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r] \arrow[d, equal] & Z \arrow[r] \arrow[d, dashed] & T(X) \arrow[d, equal] \\ X \arrow[r, "f"'] & Y \arrow[r] & Z' \arrow[r] & T(X) \end{tikzcd}$$

Thus, the two triangles are isomorphic.

$\square$

The next proposition is a criterion to characterise isomorphisms in a triangulated category.

Proposition 3.25. *A morphism $f : X \longrightarrow Y$ is an isomorphism if and only if there exists a distinguished triangle extending f and factoring through 0, that is, the triangle $X \xrightarrow{f} Y \longrightarrow 0 \longrightarrow T(X)$ is distinguished.*

Proof. Suppose f is an isomorphism. Then, the following diagram is an isomorphism of triangles:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "f"] & Y \arrow[r] \arrow[d, equal] & 0 \arrow[r] \arrow[d, equal] & T(X) \arrow[d, "T(f)"] \\ Y \arrow[r, equal] & Y \arrow[r] & 0 \arrow[r] & T(Y) \end{tikzcd}$$

By TR1, the bottom line is distinguished. Therefore, by TR0, the above triangle is distinguished as well. Conversely, suppose $X \xrightarrow{f} Y \longrightarrow 0 \longrightarrow T(X)$ is distinguished. By proposition 3.16, the sequence $Hom(K, X) \xrightarrow{f_*} Hom(K, Y) \longrightarrow 0$ is exact for every object K. In other words, $f_* = Hom(K, f)$ is surjective for all K.

Moreover, applying TR3 twice, we see that $0 \longrightarrow X \xrightarrow{f} Y \longrightarrow T(0)$ is distinguished. Again, by proposition 3.16, the sequence $0 \longrightarrow Hom(K, X) \xrightarrow{f_*} Hom(K, Y)$ is exact for every object K, giving that $f_* = Hom(K, f)$ is injective for all K. We conclude that $Hom(K, f)$ is an isomorphism for all objects K and by corollary 2.14, f is an isomorphism. $\square$

Remark 3.26. Notice the similarity to proposition 2.38 where we saw that in an abelian category a morphism $f' : X' \longrightarrow Y'$ is an isomorphism if and only if the sequence $0 \longrightarrow X' \xrightarrow{f'} Y' \longrightarrow 0$ is exact. By applying TR3 twice to the distinguished triangle considered in proposition 3.25, we see that the statement of this proposition could be rewritten as "f is an isomorphism if and only if the triangle $0 \longrightarrow X \xrightarrow{f} Y \longrightarrow 0$ is distinguished". This is another hint that distinguished triangles generalise exact sequences.

3.3 Direct sums and splittings

Recall from the definition that any triangulated category is additive. In particular, it admits finite products and coproducts, which are isomorphic by proposition 2.19. From remark 2.20, recall the notation $X \oplus Y$ to denote the object that is at the same time product and coproduct of X and Y. In this subsection, we discuss how the triangulated structure interacts with direct sums. In particular, we will give conditions for a distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X)$ to be isomorphic to a triangle of the form $X \longrightarrow X \oplus Z \longrightarrow Z \longrightarrow T(X)$. Propositions 3.27 and 3.29 come from [16], theorem 3.31 was an exercise sketched in [5].

Proposition 3.27. *Let* $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X)$ *and* $X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \xrightarrow{h'} T(X')$ *be two distinguished triangles. Then the triangle* $X \oplus X' \xrightarrow{f \oplus f'} Y \oplus Y' \xrightarrow{g \oplus g'} Z \oplus Z' \xrightarrow{h \oplus h'} T(X \oplus X')$ *is distinguished as well.*

Proof. By TR2, there exists a distinguished triangle $X \oplus X' \xrightarrow{f \oplus f'} Y \oplus Y' \longrightarrow Q \longrightarrow T(X \oplus X')$ extending $f \oplus f'$. Moreover, applying TR4 twice to the distinguished triangles we have the following rectangles:

$$\begin{tikzcd}
X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & T(X') \\
X \oplus X' \arrow[r, "f \oplus f'"'] \arrow[u, "p_{X'}"] \arrow[d, "p_X"] & Y \oplus Y' \arrow[r] \arrow[u, "p_{Y'}"] \arrow[d, "p_Y"] & Q \arrow[r] \arrow[u, dashed, "\alpha'"] \arrow[d, dashed, "\alpha"] & T(X \oplus X') \arrow[u, "T(p_{X'})"'] \arrow[d, "T(p_X)"'] \\
X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z \arrow[r, "h"'] & T(X)
\end{tikzcd}$$

Using that $Z \oplus Z' \cong Z \times Z'$, by the universal property of the product we get a map $\phi : Q \longrightarrow Z \oplus Z'$:

$$\begin{tikzcd}
 & Q \arrow[dl, "\alpha"'] \arrow[d, dashed, "\phi"'] \arrow[dr, "\alpha'"] & \\
Z & Z \oplus Z' \arrow[l, "p_Z"] \arrow[r, "p_{Z'}"'] & Z'
\end{tikzcd}$$

This leads to the following commutative diagram:

$$\begin{tikzcd}
X \oplus X' \arrow[r] \arrow[d, "id_{X \oplus X'}"'] & Y \oplus Y' \arrow[r] \arrow[d, "id_{Y \oplus Y}"'] & Q \arrow[r] \arrow[d, dashed, "\phi"'] & T(X \oplus X') \arrow[d, "id_{T(X \oplus X')}"] \\
X \oplus X' \arrow[r] & Y \oplus Y' \arrow[r] & Z \oplus Z' \arrow[r] & T(X \oplus X')
\end{tikzcd}$$

Notice that for any $K \in \mathcal{D}$, the sequence

$$Hom(K, X \oplus X') \longrightarrow Hom(K, Y \oplus Y') \longrightarrow Hom(K, Z \oplus Z') \longrightarrow Hom(K, T(X \oplus X')) \longrightarrow Hom(K, T(Y \oplus Y'))$$

is exact. Indeed, by proposition 3.16 the sequences

$$Hom(K, X) \longrightarrow Hom(K, Y) \longrightarrow Hom(K, Z) \longrightarrow Hom(K, T(X)) \longrightarrow Hom(K, T(Y))$$

and

$$Hom(K, X') \longrightarrow Hom(K, Y') \longrightarrow Hom(K, Z') \longrightarrow Hom(K, T(X')) \longrightarrow Hom(K, T(Y))$$

are exact. This implies that the sequence

$$\begin{tikzcd}
Hom(K,X) \oplus Hom(K,X') \arrow[r] & Hom(K,Y) \oplus Hom(K,Y') \arrow[r] & Hom(K,Z) \oplus Hom(K,Z') \arrow[dll] \\
Hom(K,T(X)) \oplus Hom(K,T(X')) \arrow[rr] & & Hom(K,T(Y) \oplus Hom(K,T(Y'))
\end{tikzcd}$$

is exact. By example 2.22, $Hom(K,-)$ is an additive functor, hence by proposition 2.23 it preserves direct sums and $Hom(K, X\oplus X') \cong Hom(K,X) \oplus Hom(K,X')$. Thus, we deduce exactness of the first sequence. Proposition 3.22 applies, so ϕ is an isomorphism. Therefore, by TR0,

$$X \oplus X' \xrightarrow{f\oplus f'} Y \oplus Y' \xrightarrow{g\oplus g'} Z \oplus Z' \xrightarrow{h\oplus h'} T(X\oplus X')$$

is distinguished □

Definition 3.28. We say a distinguished triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$ is split if it is isomorphic to $X \xrightarrow{\iota_X} X\oplus Z \xrightarrow{p_Z} Z \xrightarrow{w} T(X)$ where the maps are the inclusion and projection given by the universal properties of $\oplus$.

In the following, we characterise split triangles and investigate which conditions are needed so that a distinguished triangle extending some morphism f is split.

Proposition 3.29. *A distinguished triangle* $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X)$ *is split if and only if either f, g or h are the zero morphism.*

Proof. If the triangle is split consider $g' : Z \longrightarrow Y$ a section i.e $g \circ g' = id_Z$. We know this exists by composing the inclusion of Z into $X \oplus Z$ with the isomorphism between Y and $X \oplus Z$. Then, $h = h \circ id_Z = h \circ g \circ g'$. However, by proposition 3.13, $h \circ g = 0$, thus $h = 0$.
To prove the other direction, we can suppose without loss of generality (by rotating the triangle thanks to TR3 and using that T is an isomorphism), that $h = 0$.
By proposition 3.12, the triangle $0 \xrightarrow{0} Z \xrightarrow{id_Z} Z \xrightarrow{0} 0$ is distinguished and, by TR1, the triangle $X \xrightarrow{id_X} X \longrightarrow 0 \longrightarrow T(X)$ is distinguished. By proposition 3.27, this implies that

$$X \oplus 0 \longrightarrow X \oplus Z \longrightarrow 0 \oplus Z \xrightarrow{0} T(X) \oplus 0$$

is a distinguished triangle. It can be rewritten as $X \longrightarrow X \oplus Z \longrightarrow Z \xrightarrow{0} T(X)$. Consider the following diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "id_X"'] & X \oplus Z \arrow[r] \arrow[d, dashed, "\alpha"'] & Z \arrow[r, "0"] \arrow[d, "id_Z"'] & T(X) \arrow[d, "id_{T(X)}"] \\
X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z \arrow[r, "h=0"'] & T(X)
\end{tikzcd}$$

We know that α exists by remark 3.7 and by proposition 3.22 it is an isomorphism. Thus, the triangle is split. □

Corollary 3.30. *A distinguished triangle* $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X)$ *is split if and only if f admits a retract i.e. there exists $r : Y \longrightarrow X$ such that $r \circ f = id_X$ if and only if g admits a section i.e. there exists $s : Z \longrightarrow Y$ such that $g \circ s = id_Z$.*

Proof. It is clear that if the triangle is split, both the retract and the section exist. Indeed, consider the isomorphism $\varphi : X \oplus Z \longrightarrow Y$, the projection $p_X : X \oplus Z \longrightarrow X$ and the inclusion $\iota_Z : Z \longrightarrow X \oplus Z$. Setting $r = p_X \circ \varphi^{-1}$ and $s = \varphi \circ \iota_Z$ gives the desired maps.
In the proof of proposition 3.29, we already argued that if a section exists, $h = 0$ and thus, by that same proposition, the triangle is split.
Similarly, if a retraction exists, consider the rotated triangle $T^{-1}(Z) \xrightarrow{-T^{-1}(h)} X \xrightarrow{f} Y \xrightarrow{g} Z$. Then, $-T^{-1}(h) = id_X \circ -T^{-1}(h) = -r \circ -f \circ -T^{-1}(h)$, but $-f \circ -T^{-1}(h) = 0$, so $-T^{-1}(h) = 0$. This implies $h = 0$ since T is an automorphism ($h = T(T^{-1}(h)) = T(0) \cong 0$). By proposition 3.29, the triangle is split. □

Compare this characterisation of split distinguished triangles with that of short exact sequences given in proposition 2.40. In both settings, the object is split if and only if either a retraction or a splitting exist. However, the next result will show that for distinguished triangles even more is true.

Theorem 3.31. *In a triangulated category, all monomorphisms are split. In other words, any triangle extending a monomorphism splits. In particular, in view of corollary 3.30, any monomorphism admits a retraction (left-inverse).*

Proof. Let $f : X \longrightarrow Y$ be a monomorphism, let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X)$ be a distinguished triangle extending f. By TR3 and using proposition 3.13, we get that $T^{-1}(h) \circ f = 0$. As f is a monomorphism, this implies $T^{-1}(h) = 0$ and since T is an automorphism, $h = 0$. By proposition 3.29, the triangle splits. $\square$

Remark 3.32. This fails for short exact sequences in the abelian setting. Consider for example the category of $\mathbb{Z}$-modules and the short exact sequence $0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$. By corollary 2.37, $\cdot 2$ is a monomorphism. However, this sequence cannot split. Else, we would have an isomorphism $\mathbb{Z} \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. This is impossible as $(0, 1) \in \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ is an element of order 2, but $\mathbb{Z}$ contains no elements of order 2.

Not surprisingly, we get a dual statement for epimorphisms. This can be seen directly from the fact that an epimorphism is a monomorphism in the opposite category, which remains triangulated. We still include a proof however, where we present a different argument.

Theorem 3.33. *In a triangulated category, all epimorphisms split. More precisely, any distinguished triangle* $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$ *extending an epimorphism u is isomorphic to the distinguished triangle* $T^{-1}(Z) \oplus Y \xrightarrow{p_Y} Y \xrightarrow{0} Z \xrightarrow{\iota_Z} Z \oplus T(Y)$. *In particular, in view of corollary 3.30, any epimorphism admits a section (right-inverse).*

Proof. Let $u : X \longrightarrow Y$ be an epimorphism, let $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$ be a distinguished triangle extending u. This exists by TR2. By proposition 3.18, for any object $K \in \mathcal{D}$ the sequence $Hom(X, K) \xleftarrow{u^*} Hom(Y, K) \xleftarrow{v^*} Hom(Z, K)$ is exact. But u is an epimorphism meaning u^* is injective i.e. $0 = Ker(u^*) = Im(v^*)$. Thus, $v = 0$, as in particular setting $K = Z$, $id_Z \circ v = v \in Im(v^*)$. Thus the triangle has a zero-map. By proposition 3.29, this is equivalent to the triangle being split.
More precisely, we get the following diagram, where, as seen in the proof of proposition 3.29, both lines are distinguished:

$$\begin{tikzcd}
T^{-1}(Z) \arrow[r, "T^{-1}(w)"] \arrow[d, equal] & X \arrow[r, "u"] \arrow[d, dashed] & Y \arrow[r, "v=0"] \arrow[d, equal] & Z \arrow[d, equal] \\
T^{1}(Z) \arrow[r, "\iota_{T^{-1}(Z)}"] & T^{-1}(Z) \oplus Y \arrow[r, "p_Y"] & Y \arrow[r, "0"] & Z
\end{tikzcd}$$

By remark 3.23, the dashed arrow exists and is an isomorphism. Rotating the triangles gives the wanted claim. $\square$

Corollary 3.34. *In a triangulated category, a morphism $f : X \longrightarrow Y$ is an isomorphism if and only if it is both a monomorphism and an epimorphism.*

Proof. It is straightforward that an isomorphism is both a monomorphism and an epimorphism. For the converse, let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X)$ be a distinguished triangle extending f. By theorem 3.31, since f is a monomorphism, this triangle is isomorphic to the distinguished triangle $X \xrightarrow{\iota_X} X \oplus Z \xrightarrow{p_Z} Z \xrightarrow{0} T(X)$.
Using the fact that f is also an epimorphism, by theorem 3.33, the triangle extending f is also isomorhic to $T^{-1}(Z) \oplus Y \xrightarrow{p_Y} Y \xrightarrow{0} Z \xrightarrow{\iota_Z} Z \oplus T(Y)$. Explicitly, we have the following commutative diagram, where all vertical arrows are isomorphisms:

$$\begin{tikzcd}
X \arrow[r, "\iota_X"] \arrow[d, equal] & X \oplus Z \arrow[r, "p_Z"] \arrow[d, "\alpha"'] & Z \arrow[r, "0"] \arrow[d, equal] & T(X) \arrow[d, equal] \\
X \arrow[r, "f"] \arrow[d, "\beta"'] & Y \arrow[r, "g"] \arrow[d, equal] & Z \arrow[r, "h=0"] \arrow[d, equal] & T(X) \arrow[d, "T(\beta)"'] \\
T^{-1}(Z) \oplus Y \arrow[r, "p_Y"'] & Y \arrow[r, "0"'] & Z \arrow[r, "\iota_Z"'] & Z \oplus T(Y)
\end{tikzcd}$$

Thus, $p_Z = 0 \circ \alpha = 0$ and $\iota_Z = T(\beta) \circ 0 = 0$. This implies $Z = 0$. Therefore, the distinguished triangle extending f factors through 0 which implies, by proposition 3.25, that f is an isomorphism. □

Again, compare this to abelian categories. In proposition 2.38, we had shown the same statement for abelian categories. Let us sum up all the similarities between exact sequences and distinguished triangles that we encountered so far:

- The axiom TR5 makes intuitive sense if we see the distinguished triangles as short exact sequences. In this case, as seen in remark 3.5, it is a complicated way to rewrite the third isomorphism theorem for modules.
- As seen in proposition 3.11, the distinguished triangles in $\mathbf{Vect}_k$ are precisely exact sequences and by proposition 3.19 exact sequences are the only way to endow $\mathbf{Vect}_k$ with a triangulation.
- Just like in exact sequences, the composition of two adjacent morphisms in a distinguished triangle is 0 as shown in proposition 3.13.
- By proposition 3.16 and 3.18, the functors $Hom(K, -)$ and $Hom(-, K)$ turn distinguished triangles into short exact sequences and, as explained in remark 3.17, they give rise to long exact sequences.
- As illustrated by propositions 3.25 and 2.38, there is a similar characterisation of isomorphisms in an abelian category via exact sequences and of isomorphisms in a triangulated category via distinguished triangles.
- The conditions for distinguished triangles to split are very similar to those for short exact sequences to split as made clear in corollary 3.30 and proposition 2.40.
- In both abelian and triangulated categories, being an isomorphism is equivalent to being a monomorphism and an epimorphism. Moreover, in both cases the proofs use similar arguments.

However, in section 6, we will see that triangulated categories are very rarely abelian and that abelian categories are very rarely triangulated. So it really seems like the right approach is to think of distinguished triangles as generalisation of exact sequences to non-abelian settings.

3.4 Independence of the axioms

It is now time to return to the question of the independence of the axioms from the definition. When Verdier defined the triangulated category, he did not verify that the axioms were independent and it turned out they are not. Following the arguments explained in [4], we will prove that TR4 is implied by TR2 and TR5 and that the "if" or "only if" direction of TR3 is implied by the other axioms. Whether there are other dependencies among the axioms, in particular if TR5 is implied by TR0-TR4 is still an open problem today.

Proposition 3.35 (TR4)**.** *TR4 is implied by the other axioms (specifically TR2 and TR5).*

Proof. We will first prove TR4 in the two easier cases where one of the morphisms is the identity. So suppose $X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow T(X)$ and $X \xrightarrow{f'} Y' \xrightarrow{g'} Z' \longrightarrow T(X)$ are distinguished triangles and there exists a morphism $k : Y \longrightarrow Y'$ such that $k \circ f = f'$. We would like to get the existence of a morphism m_1 such that:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "g"] \arrow[d, "k"'] & Z \arrow[r] \arrow[d, dashed, "m_1"'] & T(X) \arrow[d] \\
X \arrow[r, "f'"'] & Y' \arrow[r, "g'"'] & Z' \arrow[r, equal] & T(X)
\end{tikzcd}$$

Notice that, by TR2, k can be completed to a distinguished triangle $Y \xrightarrow{k} Y' \xrightarrow{l} X' \longrightarrow T(Y)$.

Then, by TR5, the following diagram commutes and the bottom line is distinguished. This gives the existence of m_1 as wanted.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "g"] \arrow[d, "k"'] & Z \arrow[r] \arrow[d, dashed, "\exists m_1"] & T(X) \arrow[d, equal] \\
X \arrow[r, "f'"'] \arrow[d, "f"'] & Y' \arrow[r, "g'"'] \arrow[d, equal] & Z' \arrow[r] \arrow[d, dashed, "\exists m_2"] & T(X) \arrow[d, "T(f)"] \\
Y \arrow[r, "k"'] \arrow[d, "g"'] & Y' \arrow[r, "l"'] \arrow[d, "g'"'] & X' \arrow[r] \arrow[d, equal] & T(Y) \arrow[d, "T(g)"] \\
Z \arrow[r, dashed, "m_1"'] & Z' \arrow[r, dashed, "m_2"'] & X' \arrow[r, dashed] & T(Z)
\end{tikzcd}$$

Similarly, we can prove the existence of m_2 in the following diagram:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "j"'] & Y \arrow[r, "g"] \arrow[d, equal] & Z \arrow[r] \arrow[d, dashed, "m_2"'] & T(X) \arrow[d, "T(j)"] \\
X' \arrow[r, "f'"'] & Y \arrow[r, "g'"'] & Z' \arrow[r] & T(X')
\end{tikzcd}$$

By TR2, we can extend j to a distinguished triangle $X \xrightarrow{j} X' \xrightarrow{l} \tilde{Z} \longrightarrow T(X)$.
By TR5:

$$\begin{tikzcd}
X \arrow[r, "j"] \arrow[d, equal] & X' \arrow[r, "l"] \arrow[d, "f'"'] & \tilde{Z} \arrow[r] \arrow[d, dashed, "\exists u"] & T(X) \arrow[d, equal] \\
X \arrow[r, "f"'] \arrow[d, "j"'] & Y \arrow[r, "g"'] \arrow[d, equal] & Z \arrow[r] \arrow[d, dashed, "\exists m_2"] & T(X) \arrow[d, "T(j)"] \\
X' \arrow[r, "f'"'] \arrow[d, "l"'] & Y \arrow[r, "g'"'] \arrow[d, "g"'] & Z' \arrow[r] \arrow[d, equal] & T(X') \arrow[d, "T(l)"] \\
\tilde{Z} \arrow[r, dashed, "u"'] & Z \arrow[r, dashed, "m_2"'] & Z' \arrow[r, dashed] & T(\tilde{Z})
\end{tikzcd}$$

In the general case, notice that

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "j"'] & Y \arrow[r, "g"] \arrow[d, "k"] & Z \arrow[r] \arrow[d, dashed] & T(X) \arrow[d, "T(j)"] \\
X' \arrow[r, "f'"'] & Y' \arrow[r, "g'"'] & Z' \arrow[r] & T(X')
\end{tikzcd} \tag{17}$$

factors as

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "g"] \arrow[d, "k"] & Z \arrow[r] \arrow[d, dashed, "m_1"] & T(X) \arrow[d, equal] \\
X \arrow[r, "k\circ f=f'\circ j"] \arrow[d, "j"'] & Y' \arrow[r] \arrow[d, equal] & W \arrow[r] \arrow[d, dashed, "m_2"] & T(X) \arrow[d, "T(j)"] \\
X' \arrow[r, "f'"'] & Y' \arrow[r, "g'"] & Z' \arrow[r] & T(X')
\end{tikzcd}$$

where the middle triangle is extending $k \circ f = f' \circ j$ given by TR2. Thus, the dashed arrow in (17) can be completed with $m_2 \circ m_1$. $\square$

Proposition 3.36 (TR3)**.** *The "only if" part of TR3 is implied by the other axioms (including the "if" part of TR3). In other words, if* $Y \xrightarrow{-g} Z \xrightarrow{-h} T(X) \xrightarrow{-T(f)} T(Y)$ *is distinguished, then* $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X)$ *is distinguished as well.*

Proof. By TR2, there exists a distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g'} W \xrightarrow{h'} T(X)$ extending f. Applying the "if" statement of TR3 three times, it follows that $T(X) \xrightarrow{-T(f)} T(Y) \xrightarrow{-T(g')} T(W) \xrightarrow{-T(h')} T(T(X))$ is distinguished. Similarly, applying the "if" statement of TR3 twice to $Y \xrightarrow{-g} Z \xrightarrow{-h} T(X) \xrightarrow{-T(f)} T(Y)$ and by remark 3.3, we find that the triangle $T(X) \xrightarrow{-T(f)} T(Y) \xrightarrow{-T(g)} T(Z) \xrightarrow{-T(h)} T(T(X))$ is distinguished. Thus, we have the following diagram:

$$\begin{tikzcd}
T(X) \arrow[r, "-T(f)"] \arrow[d, equal] & T(Y) \arrow[r, "-T(g')"] \arrow[d, equal] & T(W) \arrow[r, "-T(h')"] \arrow[d, dashed, "\gamma"] & T(T(X)) \arrow[d, equal] \\
T(X) \arrow[r, "-T(f)"'] & T(Y) \arrow[r, "-T(g)"'] & T(Z) \arrow[r, "-T(h)"'] & T(T(X))
\end{tikzcd}$$

The morphism γ exists by TR4 and is an isomorphism by proposition 3.22. Notice that in both the proof of TR4 and of proposition 3.22, we haven't used the "only if" direction of TR3. Since T^{-1} takes isomorphisms to isomorphisms, we can now apply $-T^{-1}$ to the diagram, in order to get an isomorphism of triangles between $X \xrightarrow{f} Y \xrightarrow{g'} W \xrightarrow{h'} T(X)$ and $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X)$. Since the first triangle is distinguished, we conclude by TR0 that the second is too. $\square$

In view of those propositions, we won't verify TR4 and one direction of TR3 when proving that a given category is triangulated from now on.

4 The Homotopy Category $K_d(\mathcal{A})$

In this section, we prove one of the most classical results on triangulated categories: that the homotopy category $K_d(\mathcal{A})$ of an additive category $\mathcal{A}$ is triangulated. We begin by introducing the necessary definitions and end this section by presenting the proof of this result. We follow the notations and arguments from [9], but some ideas are also inspired by [8].

Definition 4.1. Let $(\mathcal{A}, T)$ be an additive category with translation. A priori, it is unclear whether for a given object $X \in \mathcal{A}$ there exists a morphism from X to $T(X)$. If there is a map $d_X : X \longrightarrow T(X)$, we say that (X, d_X) is a differential object in $(\mathcal{A}, T)$. The map d_X is called a differential. $T(X)$ is a shifted object and becomes a differential object when we endow it with $d_{T(X)} = -T(d_X)$. It can happen that there exist several differentials $d_X : X \longrightarrow T(X)$ and $d'_X : X \longrightarrow T(X)$. Then, (X, d_X) and (X, d'_X) are two different differential objects. Still, by abuse of notation, we often denote the differential object only by X.
A morphism $f : X \longrightarrow Y$ between two differential objects (X, d_X) and (Y, d_Y) is said to be a morphism of differential objects if the following square commutes.

$$\begin{tikzcd} X \arrow[r, "d_X"] \arrow[d, "f"'] & T(X) \arrow[d, "T(f)"] \\ Y \arrow[r, "d_Y"'] & T(Y) \end{tikzcd}$$

We denote by $\mathcal{A}_d$ the subcategory of $\mathcal{A}$ of differential objects.
A complex is a differential object (X, d_X) such that $T(d_X) \circ d_X = 0$. We denote by $\mathcal{A}_c$ the subcategory of $\mathcal{A}$ of complexes. Notice that the shifted object of a complex is still a complex.

Remark 4.2. Observe that if $\mathcal{A}$ is an abelian category, then so are A_d and A_c. This follows directly from the observation that the kernel, image and cokernel of a morphism of differential objects are differential objects.

A reader who is familiar with homological algebra has certainly already encountered the notion of chain complex and might now wonder whether the definition of complex object given here and the usual definition of chain complex coincide. As will be explained in the following example, this is indeed the case. The definition given here is nothing more than a generalisation of the usual definition.

Example 4.3. Consider an additive category $\mathcal{A}$. A common choice for $\mathcal{A}$ in algebraic topology is for example $\mathcal{A} = Ab$, the category of abelian groups. We define the category of $\mathbb{Z}$-graded objects of $\mathcal{A}$, denoted $Gr(\mathcal{A})$, as the category whose objects are $\mathbb{Z}$-indexed families of objects $X^i \in \mathcal{A}$, denoted $\{X^i\}_{i\in\mathbb{Z}}$ and whose morphisms are $\mathbb{Z}$-indexed families of morphisms in $\mathcal{A}$ denoted by $\{f_i\}_{i\in\mathbb{Z}}$. The collection of morphisms between two objects is given by

$$Hom_{Gr(\mathcal{A})}(\{X^i\}_{i\in\mathbb{Z}}, \{Y^i\}_{i\in\mathbb{Z}}) = \{\{f_i\}_{i\in\mathbb{Z}} \mid \forall i \ f_i : X^i \longrightarrow Y^i\}.$$

Composition and the identities are defined in the obvious way. It is easy to verify that this category remains additive. We will often write X as shorthand for an object $\{X^i\}_{i\in\mathbb{Z}} \in Gr(\mathcal{A})$. If we want to emphasise that it is a graded object we write $X_\bullet$. The same notation also applies to morphisms. We define a translation functor $T : Gr(\mathcal{A}) \to Gr(\mathcal{A})$ by $T(\{X^i\}_{i\in\mathbb{Z}}) = \{X^{i+1}\}_{i\in\mathbb{Z}}$ and $T(\{f_i\}_{i\in\mathbb{Z}}) = \{f_{i+1}\}_{i\in\mathbb{Z}}$.
Then, $Gr(\mathcal{A})_d$, the subcategory of differential objects of $Gr(\mathcal{A})$, consists of all those objects $\{X^i\}_{i\in\mathbb{Z}}$ such that there is a morphism $\{d_i\}_{i\in\mathbb{Z}} : \{X^i\}_{i\in\mathbb{Z}} \longrightarrow \{X^{i+1}\}_{i\in\mathbb{Z}}$. Thus, the differential objects of $Gr(\mathcal{A})$ can be visualised as the following diagram:

$$\cdots \xrightarrow{d_{i-2}} X^{i-1} \xrightarrow{d_{i-1}} X^i \xrightarrow{d_i} X^{i+1} \xrightarrow{d_{i+1}} \cdots \tag{18}$$

T can be visualised as the functor that shifts everything to the left by one.
A morphism $\{\phi_i\}_{i\in\mathbb{Z}} : \{X^i\}_{i\in\mathbb{Z}} \longrightarrow \{Y^i\}_{i\in\mathbb{Z}}$ of $Gr(\mathcal{A})_d$ needs to be a morphism of differential objects. Writing d for the differential of X and δ for the differential of Y, this means that $\delta \circ \phi = T(\phi) \circ d$. The equality has to hold at each index, so writing out the indices, we have $\delta_i \circ \phi_i = \phi_{i+1} \circ d_i \ \forall i \in \mathbb{Z}$. In terms of diagrams, this means that the following diagram commutes:

$$\begin{tikzcd} \cdots \arrow[r, "d_{i-2}"] & X^{i-1} \arrow[r, "d_{i-1}"] \arrow[d, "\phi_{i-1}"'] & X^i \arrow[r, "d_i"] \arrow[d, "\phi_i"'] & X^{i+1} \arrow[r, "d_{i+1}"] \arrow[d, "\phi_{i+1}"'] & \cdots \\ \cdots \arrow[r, "\delta_{i-2}"] & Y^{i-1} \arrow[r, "\delta_{i-1}"] & Y^i \arrow[r, "\delta_i"] & Y^{i+1} \arrow[r, "\delta_{i+1}"] & \cdots \end{tikzcd}$$

We retrieve the familiar notion of chain map.
Furthermore, $Gr(\mathcal{A})_c$, the subcategory of complexes of $Gr(\mathcal{A})$, consists of all the differential objects such that $T(d_i) \circ d_i = d_{i+1} \circ d_i = 0$, that is diagrams like (18) with the additional condition on d. Thus, we recover indeed the familiar definition of chain complex!
Notice that if $\mathcal{A}$ is abelian, then $Gr(\mathcal{A})$ is abelian and thus, by remark 4.2, $Gr(\mathcal{A})_d$ and $Gr(\mathcal{A})_c$ are abelian categories as well.

In order to be able to define the distinguished triangles, we need to introduce the notions of mapping cone and mapping cone triangle. In the following, we fix an additive category with translation $(\mathcal{A}, T)$.

Definition 4.4 (mapping cone)**.** For a morphism of differential objects $f : X \longrightarrow Y$ between two differential objects in $\mathcal{A}$, we define the mapping cone of f, $Mc(f)$, as the differential object given by $(T(X) \oplus Y, d_{Mc(f)})$ in $\mathcal{A}$. The differential is given by $d_{Mc(f)} = \begin{pmatrix} d_{T(X)} & 0 \\ T(f) & d_Y \end{pmatrix}$.

It is important to realise that, although as objects of $\mathcal{A}$ the mapping cone $Mc(f)$ is the same as the direct sum $T(X) \oplus Y$, these two objects differ in $\mathcal{A}_d$ (seen as differential objects). Indeed, they are equal as differential objects if and only if the differentials $d_{Mc(f)}$ and $d_{T(X)\oplus Y} := \begin{pmatrix} d_{T(X)} & 0 \\ 0 & d_Y \end{pmatrix}$ coincide. This happens if and only if f is the zero morphism.

Example 4.5. In the setting of the previous example, consider:

$$\begin{tikzcd}
\cdots \arrow[r, "d_{i-2}"] & X^{i-1} \arrow[r, "d_{i-1}"] \arrow[d, "\phi_{i-1}"'] & X^i \arrow[r, "d_i"] \arrow[d, "\phi_i"'] & X^{i+1} \arrow[r, "d_{i+1}"] \arrow[d, "\phi_{i+1}"'] & \cdots \\
\cdots \arrow[r, "\delta_{i-2}"] & Y^{i-1} \arrow[r, "\delta_{i-1}"] & Y^i \arrow[r, "\delta_i"] & Y^{i+1} \arrow[r, "\delta_{i+1}"] & \cdots
\end{tikzcd}$$

where X, Y are chain complexes and all the squares commute, i.e ϕ is a chain map. The mapping cone of ϕ is the chain complex:

$$\cdots \longrightarrow X^i \oplus Y^{i-1} \xrightarrow{\begin{pmatrix} -d_i & 0 \\ \phi_i & \delta_{i-1} \end{pmatrix}} X^{i+1} \oplus Y^i \xrightarrow{\begin{pmatrix} -d_{i+1} & 0 \\ \phi_{i+1} & \delta_i \end{pmatrix}} X^{i+2} \oplus Y^{i+1} \longrightarrow \cdots$$

Definition 4.6 (mapping cone triangle)**.** The mapping cone triangle of a morphism $f : X \longrightarrow Y$ is the triangle

$$X \xrightarrow{f} Y \xrightarrow{\alpha(f)} Mc(f) \xrightarrow{\beta(f)} T(X)$$

where $\alpha(f) = 0 \oplus id_Y$ is the inclusion of Y in the mapping cone and $\beta(f) = \begin{pmatrix} id_{T(X)} & 0 \end{pmatrix}$ the projection onto $T(X)$. Observe that, if $\mathcal{A}$ is abelian, this gives rise to a short exact sequence $0 \longrightarrow Y \overset{\alpha(f)}{\longrightarrow} Mc(f) \overset{\beta(f)}{\longrightarrow} X \longrightarrow 0$.

We introduce a tool to compare morphisms of differential objects, the homotopy. We will see that identifying morphism up to homotopy is crucial in order to get a triangulated structure on the differential objects.

Definition 4.7. A morphism of differential objects $f : X \longrightarrow Y$ is homotopic to 0 if there exists a morphism $u : X \longrightarrow T^{-1}(Y)$ in $\mathcal{A}$, referred to as homotopy, such that $f = T(u) \circ d_X + T^{-1}(d_Y) \circ u$. Two morphisms f, g are homotopic if $f - g$ is homotopic to zero.
We visualise being homotopic to zero via the following non-commutative diagram:

$$\begin{tikzcd}
T^{-1}(X) \arrow[r] & X \arrow[r, "d_X"] \arrow[dl, "u"'] \arrow[d, "f"'] & T(X) \arrow[dl, "T(u)"] \\
T^{-1}(Y) \arrow[r, "T^{-1}(d_Y)"'] & Y \arrow[r, "d_Y"'] & T(Y)
\end{tikzcd}$$

Example 4.8. ϕ from the previous example is homotopic to zero if there exists some morphism $u : X \to T^{-1}(Y)$ such that $\phi_i = \delta_{i-1} \circ u_i + u_{i+1} \circ d_i \ \forall i$. Notice that we do not require that u is a morphism of chain complexes, i.e. that $\delta_{i-1} \circ u_i = u_{i+1} \circ d_i$. Such a condition would actually be quite restrictive as then $\phi_i = 2(\delta_{i-1} \circ u_i)$ would have to factor by 2. In this setting, being homotopic to zero is visualised by the following non-commutative diagram, which might be familiar to an algebraic topologist:

$$\begin{tikzcd}
\cdots \arrow[r, "d_{i-2}"] & X^{i-1} \arrow[r, "d_{i-1}"] \arrow[dl, dashed, "u_{i-1}"'] \arrow[d, "\phi_{i-1}"'] & X^i \arrow[r, "d_i"] \arrow[dl, dashed, "u_i"'] \arrow[d, "\phi_i"'] & X^{i+1} \arrow[r, "d_{i+1}"] \arrow[dl, dashed, "u_{i+1}"'] \arrow[d, "\phi_{i+1}"'] & \cdots \arrow[dl, dashed] \\
\cdots \arrow[r, "\delta_{i-2}"'] & Y^{i-1} \arrow[r, "\delta_{i-1}"'] & Y^i \arrow[r, "\delta_i"'] & Y^{i+1} \arrow[r, "\delta_{i+1}"'] & \cdots
\end{tikzcd}$$

The following proposition shows that identifying morphisms up to homotopy behaves well with composition. This will be useful to define the homotopy category.

Proposition 4.9. *Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be morphisms of differential objects. If either f or g is homotopic to zero, then the composition $g \circ f$ is homotopic to zero as well.*

Proof. Suppose first that f is homotopic to 0 i.e. there exists a morphism $u : X \longrightarrow T^{-1}(Y)$ such that $f = T(u) \circ d_X + T^{-1}(d_Y) \circ u$. As g is a morphism of differential objects, it holds:

$$g\circ f = g\circ T(u)\circ d_X + g\circ T^{-1}(d_Y)\circ u = g\circ T(u)\circ d_X + T^{-1}(d_Z)\circ T^{-1}(g)\circ u = T(T^{-1}(g)\circ u)\circ d_X + T^{-1}(d_Z)\circ T^{-1}(g)\circ u.$$

Thus, $g \circ f$ is homotopic to zero, the homotopy is given by $T^{-1}(g) \circ u$. The other case is proved similarly. $\square$

We are at last ready to define the homotopy category. Set $Ht(X,Y) = \{f \in Hom_{\mathcal{A}_d}(X,Y) \mid f \text{ is homotopic to } 0\}$. Recall that $\mathcal{A}_d$ is an additive category, so $Hom_{\mathcal{A}_d}(X,Y)$ is a $\mathbb{Z}$-module. It is easy to verify that $Ht(X,Y)$ is an additive subgroup of $Hom_{\mathcal{A}_d}(X,Y)$. Thus the quotient $Hom_{\mathcal{A}_d}(X,Y)/Ht(X,Y)$ is well-defined. By proposition 4.9, composition in $\mathcal{A}_d$ induces a bilinear map by passing to the quotient:

$$\begin{aligned} Hom_{\mathcal{A}_d}(X,Y)/Ht(X,Y) \times Hom_{\mathcal{A}_d}(Y,Z)/Ht(Y,Z) &\longrightarrow Hom_{\mathcal{A}_d}(X,Z)/Ht(X,Z) \\ ([f],[g]) &\mapsto [g \circ f] \end{aligned} \tag{19}$$

where $[f]$ and $[g]$ denote the equivalence classes in the quotient of a morphism $f \in Hom_{\mathcal{A}_d}(X,Y)$ and $g \in Hom_{\mathcal{A}_d}(Y,Z)$.

Definition 4.10. The homotopy category of $\mathcal{A}$, $K_d(\mathcal{A})$, is defined by:

$$Ob(K_d(\mathcal{A})) = Ob(\mathcal{A}_d),$$

$$Hom_{K_d(\mathcal{A})}(X,Y) = Hom_{\mathcal{A}_d}(X,Y)/Ht(X,Y)$$

The composition of morphisms is given by (19).

In other words, in $K_d(A)$, we identify all homotopic morphisms. Morphisms homotopic to zero become the zero morphism. A morphism of differential objects $f : X \to Y$ is an isomorphism in $K_d(\mathcal{A})$ if there exists a morphism $g : Y \to Z$ such that $g \circ f$ is homotopic to id_X and $f \circ g$ is homotopic to id_Y. The quotient $Hom_{\mathcal{A}_d}(X,Y)/Ht(X,Y)$ is the same as the quotient by the equivalence relation $Hom_{\mathcal{A}_d}(X,Y)/\sim$ where $f \sim g$ if and only if f is homotopic to g. It is straightforward that this defines an equivalence relation. It is easy to check that $K_d(\mathcal{A})$ remains additive and the translation functor T of $\mathcal{A}_d$ induces a translation functor on $K_d(\mathcal{A})$, which we will also denote T.

Theorem 4.11. *$(K_d(\mathcal{A}), T)$ is a triangulated category. Distinguished triangles are given by those isomorphic (in $K_d(\mathcal{A})$) to a mapping cone triangle.*

Proof. We check the axioms from the definition of triangulated category one by one. TR0 and TR2 hold by definition of the triangulation of $(K_d(\mathcal{A}), T)$. Thanks to proposition 3.35, there's no need to check TR4 and by proposition 3.36 we only check one direction of TR3. We begin with TR3, then verify TR1 and finish with the proof of TR5. Moreover, without loss of generality, it is sufficient to show all the axioms only for mapping cone triangles.

TR3: Consider some distinguished triangle $X \xrightarrow{f} Y \xrightarrow{\alpha(f)} Mc(f) \xrightarrow{\beta(f)} T(X)$. We want the triangle $Y \xrightarrow{\alpha(f)} Mc(f) \xrightarrow{\beta(f)} T(X) \xrightarrow{-T(f)} T(Y)$ to be distinguished as well, that is it has to be isomorphic to the mapping cone triangle of $\alpha(f)$. In other words, we need an isomorphism ϕ of $K_d(\mathcal{A})$ that completes:

$$\begin{tikzcd} Y \arrow[r, "\alpha(f)"] \arrow[d, equal] & Mc(f) \arrow[r, "\beta(f)"] \arrow[d, equal] & T(X) \arrow[r, "-T(f)"] \arrow[d, dashed, "\phi"'] & T(Y) \arrow[d, equal] \\ Y \arrow[r, "\alpha(f)"'] & Mc(f) \arrow[r, "\alpha(\alpha(f))"'] & Mc(\alpha(f)) \arrow[r, "\beta(\beta(f))"'] & T(Y) \end{tikzcd} \tag{20}$$

Recall from the definition that $Mc(f) = T(X) \oplus Y$ and its differential is $d_{Mc(f)} = \begin{pmatrix} -T(d_X) & 0 \\ T(f) & d_Y \end{pmatrix}$.
Hence, by definition, $Mc(\alpha(f)) = T(Y) \oplus Mc(f) = T(Y) \oplus T(X) \oplus Y$ and the differential $d_{Mc(\alpha(f))}$ is given by $\begin{pmatrix} -T(d_Y) & 0 \\ T(\alpha(f)) & d_{Mc(f)} \end{pmatrix} = \begin{pmatrix} -T(d_Y) & 0 & 0 \\ 0 & -T(d_X) & 0 \\ T(id_Y) & T(f) & d_Y \end{pmatrix}$.

We define $\phi : T(X) \longrightarrow Mc(\alpha(f))$ by $\phi = \left(\begin{smallmatrix} -T(f) \\ id_{T(X)} \\ 0 \end{smallmatrix}\right)$ and $\psi : Mc(\alpha(f)) \longrightarrow T(X)$ by $\psi = \left(\begin{smallmatrix} 0 & id_X & 0 \end{smallmatrix}\right)$.

Remembering that f is a morphism of differential objects, it is straightforward to check that ϕ and ψ are also morphisms of differential objects. We will now verify that (20) commutes with these choices of ϕ and ψ and we check that they are each other's inverse, that is we check that $\phi \circ \psi$ is homotopic to $id_{Mc(\alpha(f))}$ and $\psi \circ \phi$ is homotopic to $id_{T(X)}$.

It is obvious from the definition that $\psi \circ \phi = id_{T(X)}$, $\psi \circ \alpha(\alpha(f)) = \beta(f)$ and $\beta(\alpha(f)) \circ \phi = -T(f)$. To conclude, it remains to prove that $\phi \circ \psi$ is homotopic to $id_{Mc(\alpha(f))}$ and that $\phi \circ \beta(f)$ is homotopic to $\alpha(\alpha(f))$. The latter will follow from the former since $\phi \circ \beta(f) = \phi \circ \psi \circ \alpha(\alpha(f))$.

So let us prove that $\phi \circ \psi$ is homotopic to $id_{Mc(\alpha(f))}$ by showing that $\phi \circ \psi - id_{Mc(\alpha(f))}$ is homotopic to 0. By definition of being homotopic to 0, we need to find a map $u : Mc(\alpha(f)) \to T^{-1}(Mc(\alpha(f)))$ such that $id_{Mc(\alpha(f))} - \phi \circ \psi = T(u) \circ d_{Mc(\alpha(f))} + T^{-1}(d_{Mc(\alpha(f))}) \circ u$.

Observe that $\phi \circ \psi = \begin{pmatrix} 0 & -T(f) & 0 \\ 0 & id_{T(X)} & 0 \\ 0 & 0 & 0 \end{pmatrix}$, so $id_{Mc(\alpha(f))} - \phi \circ \psi = \begin{pmatrix} id_{T(Y)} & T(f) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & id_Y \end{pmatrix}$.

We set $u = \begin{pmatrix} 0 & 0 & id_Y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and verify that $T(u) \circ d_{Mc(\alpha(f))} + T^{-1}(d_{Mc(\alpha(f))}) \circ u = id_{Mc(\alpha(f))} - \phi \circ \psi$.

$$T(u) \circ d_{Mc(\alpha(f))} + T^{-1}(d_{Mc(\alpha(f))}) \circ u =$$

$$\begin{pmatrix} 0 & 0 & T(id_Y) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -T(d_Y) & 0 & 0 \\ 0 & -T(d_X) & 0 \\ T(id_Y) & T(f) & d_Y \end{pmatrix} + \begin{pmatrix} -d_Y & 0 & 0 \\ 0 & -d_X & 0 \\ id_Y & f & T^{-1}(d_Y) \end{pmatrix} \begin{pmatrix} 0 & 0 & id_Y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} id_{T(y)} & T(f) & d_Y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & -d_Y \\ 0 & 0 & 0 \\ 0 & 0 & id_Y \end{pmatrix} = \begin{pmatrix} id_{T(Y)} & T(f) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & id_Y \end{pmatrix} = id_{Mc(\alpha(f))} - \phi \circ \psi.$$

Hence, u exhibits a homotopy between $\phi \circ \psi$ and $id_{Mc(\alpha(f))}$. Therefore, diagram (20) depicts an isomorphism between the mapping cone triangle of $\alpha(f)$ and $Y \xrightarrow{\alpha(f)} Mc(f) \xrightarrow{\beta(f)} T(X) \xrightarrow{-T(f)} T(Y)$. By definition of the triangulation in $K_d(\mathcal{A})$, the latter is thus distinguished, proving TR3.

TR1: TR1 follows as $0 \longrightarrow X = X \longrightarrow 0$ is a mapping cone triangle for $0 \longrightarrow X$ and by TR3 (recall that TR1 wasn't used in the proof of prop. 3.36) the triangle $X = X \longrightarrow 0 \longrightarrow T(X)$ is also distinguished.

TR5: To prove the last axiom, it is sufficient to find the dashed arrows making the last line in the following diagram a distinguished triangle:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r] \arrow[d, "g"] & Mc(f) \arrow[r] \arrow[d, dashed, "u"] & T(X) \arrow[d, equal] \\
X \arrow[r, "g\circ f"] \arrow[d, "f"'] & Z \arrow[r] \arrow[d, equal] & Mc(g\circ f) \arrow[r] \arrow[d, dashed, "v"] & T(X) \arrow[d, "T(f)"] \\
Y \arrow[r, "g"] \arrow[d] & Z \arrow[r] \arrow[d] & Mc(g) \arrow[r] \arrow[d, equal] & T(Y) \arrow[d] \\
Mc(f) \arrow[r, dashed, "u"] & Mc(g\circ f) \arrow[r, dashed, "v"] & Mc(g) \arrow[r] & T(Mc(f))
\end{tikzcd}$$

Define $u : T(X) \oplus Y \longrightarrow T(X) \oplus Z$ by $u = \begin{pmatrix} id_{T(X)} & 0 \\ 0 & g \end{pmatrix}$ and $v = T(X) \oplus Z \longrightarrow T(Y) \oplus Z$ by $v = \begin{pmatrix} T(f) & 0 \\ 0 & id_Z \end{pmatrix}$. Clearly, these choices make the diagram commutative.

We still need to check that the bottom line is distinguished. As in the proof of TR3, we need to construct an isomorphism ϕ in $K_d(\mathcal{A})$ and its inverse ψ that completes:

$$\begin{tikzcd}
Mc(f) \arrow[r, "u"] \arrow[d, equal] & Mc(g\circ f) \arrow[r, "v"] \arrow[d, equal] & Mc(g) \arrow[r, "w"] \arrow[d, dashed, "\phi"'] & T(Mc(f)) \arrow[d, equal] \\
Mc(f) \arrow[r, "u"'] & Mc(g\circ f) \arrow[r, "\alpha(u)"'] & Mc(u) \arrow[r, "\beta(u)"'] & T(Mc(f))
\end{tikzcd}$$

where w denotes the composition $T(\alpha(f)) \circ \beta(g)$.

By definition, $Mc(u) = T(Mc(f)) \oplus Mc(g\circ f) = T^2(X) \oplus T(Y) \oplus T(X) \oplus Z$ and $Mc(g) = T(Y) \oplus Z$.

Set $\psi = \begin{pmatrix} 0 & id_{T(Y)} & T(f) & 0 \\ 0 & 0 & 0 & id_Z \end{pmatrix}$ and $\phi = \begin{pmatrix} 0 & 0 \\ id_{T(Y)} & 0 \\ 0 & 0 \\ 0 & id_Z \end{pmatrix}$.

Again, it is clear that these maps are morphisms of differential objects (remembering f is), that $\psi \circ \phi = id_{Mc(g)}$, $\psi \circ \alpha(u) = v$ and $\beta(u) \circ \phi = w$. Finally, we need to check that $\phi \circ \psi$ is homotopic to $id_{Mc(u)}$, the remaining verifications for commutativity follow easily.

In order to do so, we need to show that $id_{Mc(u)} - \psi \circ \phi$ is homotopic to 0. In other words, we need to define a map $s : Mc(u) \longrightarrow T^{-1}(Mc(u))$ such that $T(s) \circ d_{Mc(u)} + T^{-1}(d_{Mc(u)}) \circ s = id_{Mc(u)} - \psi \circ \phi$.

Set $s = \begin{pmatrix} 0 & 0 & id_{T(X)} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ Then,

$$T(s) \circ d_{Mc(u)} + T^{-1}(d_{Mc(u)}) \circ s =$$

$$\begin{pmatrix} 0 & 0 & id_{T^2(X))} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} T^2(d_X) & 0 & 0 & 0 \\ -T^2(f) & -T(d_y) & 0 & 0 \\ id_{T^2(x)} & 0 & -T(d_X) & 0 \\ 0 & T(g) & T(g\circ f) & d_z \end{pmatrix} + \begin{pmatrix} T(d_X) & 0 & 0 & 0 \\ -T(f) & -d_y & 0 & 0 \\ id_{T(x)} & 0 & -d_X & 0 \\ 0 & g & g\circ f & d_z \end{pmatrix} \begin{pmatrix} 0 & 0 & id_{T(X)} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} id_{T^2(X))} & 0 & -T(d_X) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & T(d_X) & 0 \\ 0 & 0 & -T(f) & 0 \\ 0 & 0 & id_{T(X)} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} id_{T^2(X)} & 0 & 0 & 0 \\ 0 & 0 & -T(f) & 0 \\ 0 & 0 & id_{T(X)} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} id_{T^2(X)} & 0 & 0 & 0 \\ 0 & id_{T(y)} & 0 & 0 \\ 0 & 0 & id_{T(X)} & 0 \\ 0 & 0 & 0 & id_Z \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & id_{T(Y)} & T(f) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & id_Z \end{pmatrix} = id_{Mc(u)} - \psi \circ \phi.$$

$\square$

Remark 4.12. Notice that the family of distinguished triangles isomorphic in $\mathcal{A}_d$ to mapping cone triangles cannot induce a triangulated structure on $\mathcal{A}_d$. Indeed, $X = X \longrightarrow 0 \longrightarrow T(X)$ is a mapping cone triangle if and only if the mapping cone of the identity, $T(X) \oplus X$ is isomorphic to zero. This holds if and only if $X = 0$. So identifying morphisms up to homotopy was crucial to get this triangulation.

Remark 4.13. Focusing on complexes rather than differential objects, one can also define the category $K_c(\mathcal{A})$ characterised by:

$$Ob(K_c(\mathcal{A})) = Ob(\mathcal{A}_c),$$

$$Hom_{K_c(\mathcal{A})}(X,Y) = Hom_{\mathcal{A}_c}(X,Y)/Ht(X,Y)$$

This is a well-defined category by the same arguments as for $K_d(\mathcal{A})$. A similar proof as that of theorem 4.11 shows that $K_c(\mathcal{A})$ is triangulated.

Recall that in proposition 3.11 and 3.19, we had shown that in $\mathbf{Vect}_k$ distinguished triangles could only be defined as certain exact sequences. We had hinted that this was a general pattern and at the end of section 3.3 we had listed arguments that reinforced the intuition that distinguished triangles generalise exact sequences to non-abelian settings. The homotopy category provides some more insight on the relation between exact sequences and distinguished triangles. Indeed, in definition 4.6, we had observed that, if $\mathcal{A}$ is abelian, the sequence $0 \longrightarrow Y \xrightarrow{\alpha(f)} Mc(f) \xrightarrow{\beta(f)} X \longrightarrow 0$ is exact. Since any distinguished triangle in the homotopy category is isomorphic to some mapping cone triangle, we can associate an exact sequence to any distinguished triangle (the sequence mentioned above). However, it is not true in general that to any exact sequence $0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$ we can associate a triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow T(X)$ that is distinguished in the homotopy category. For example, consider the category $K_c(Gr(Ab))$. As introduced in example 4.3, the objects of this category are chain complexes of abelian groups and the morphisms are chain maps that are identified up to homotopy. The short exact sequence of abelian groups $0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$ can be seen as short exact sequence of chain complexes concentrated at degree 0. We show that there cannot be a distinguished triangle in $K_c(Gr(Ab))$ of the form $\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{w} T(\mathbb{Z}/2\mathbb{Z})$. Suppose for contradiction such a triangle existed. Since the chain complexes are concentrated at degree 0, there are no non-zero morphisms between $\mathbb{Z}/2\mathbb{Z}$ and $T(\mathbb{Z}/2\mathbb{Z})$, thus $w = 0$. In view of proposition 3.29, the triangle must be split, i.e. $\mathbb{Z}$ must be isomorphic in $K_c(Gr(Ab))$ to $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. All complexes being concentrated at degree 0, an isomorphism up to homotopy must be an isomorphism of abelian groups, which we know to be impossible (see remark 3.32). Thus, such a distinguished triangle cannot exist. This motivates the introduction of another triangulated category known as the derived category. It is constructed from the homotopy category and will have the property that any distinguished triangle corresponds to a short exact sequence and that any short exact sequence fits into a distinguished triangle. Moreover, remember from the introduction that the derived category was Verdier's historic motivation.

5 Derived category

5.1 Cohomology

In order to construct the derived category, we must introduce the notion of cohomology, in particular of the cohomology functor H. Again, we follow [9].
Let $\mathcal{A}$ be an abelian category with translation T. As before, denote $\mathcal{A}_d$ and $\mathcal{A}_c$ the subcategories of differential objects and complexes. Consider $X \in \mathcal{A}_c$. Recall we have differentials $d_X : X \longrightarrow T(X)$ and $T^{-1}(d_X) : T^{-1}(X) \longrightarrow X$ and that $d_X \circ T^{-1}(d_X) = 0$. So, by the universal property of $\ker(d_X)$, the morphism $\alpha : \mathrm{im}(T^{-1}(d_X)) \longrightarrow X$ factors uniquely through $\ker(d_X)$ i.e. there is a unique map $\alpha_X : \mathrm{im}(T^{-1}(d_X)) \longrightarrow \ker(d_X)$ such that $\alpha = h \circ \alpha_X$ where $h : \ker(d_X) \longrightarrow X$ comes with the universal property of the kernel.

Definition 5.1 (cohomology)**.** Consider an abelian category with translation $(\mathcal{A}, T)$ and let X be an object of the subcategory $\mathcal{A}_c$. The cohomology of X is an object of $\mathcal{A}$, denoted by $H(X)$, defined as $H(X)$=coker$(\alpha_X : \mathrm{im}(T^{-1}(d_X)) \longrightarrow \ker(d_X))$.

The reader might have encountered the notion of cohomology of a chain complex in the context of algebraic topology or homological algebra. As the following example shows, the above definition is a generalisation of this concept.

Example 5.2. Consider the case where $\mathcal{A} = Gr(Ab)$, the category of $\mathbb{Z}$-graded objects of Ab as defined in example 4.3. Recall that the subcategory $\mathcal{A}_c = Gr(Ab)_c$ is the category whose objects are chain complexes of abelian groups and whose morphisms are chain maps. Let X be such a chain complex:

$$\cdots \xrightarrow{d_{-1}} X^0 \xrightarrow{d_0} X^1 \xrightarrow{d_1} X^2 \xrightarrow{d_2} \cdots$$

We wish to calculate $H(X)$, which will be a $\mathbb{Z}$-graded object of abelian groups we will denote $\{H^i(X)\}_{i\in\mathbb{Z}}$. Recall that $T^{-1}(X)$ is the same complex, but X^i is now at degree $i+1$, that is, we shifted everything to the right. We have the following chain map between $T^{-1}(X)$ and X.

$$\begin{tikzcd}
\cdots \arrow[r, "d_{-2}"] & X^{-1} \arrow[r, "d_{-1}"] \arrow[d, dashed, "d_{-1}"] & X^0 \arrow[r, "d_0"] \arrow[d, dashed, "d_0"] & X^1 \arrow[r, "d_1"] \arrow[d, dashed, "d_1"] & \cdots \\
\cdots \arrow[r, "d_{-1}"] & X^0 \arrow[r, "d_0"] & X^1 \arrow[r, "d_1"] & X^2 \arrow[r, "d_2"] & \cdots
\end{tikzcd}$$

Since the composition $d_{i+1} \circ d_i = 0$, the chain map factors through $\ker(d_X)$. The situation is illustrated in the following diagram:

$$\begin{tikzcd}
\cdots \arrow[r, "d_{-2}"] & X^{-1} \arrow[r, "d_{-1}"] \arrow[d, "d_{-1}"'] & X^0 \arrow[r, "d_0"] \arrow[d, "d_0"'] & X^1 \arrow[r, "d_1"] \arrow[d, "d_1"'] & \cdots \\
\cdots \arrow[r] & \mathrm{im}(d_{-1}) \arrow[r] \arrow[d, "\alpha_{X^0}"'] & \mathrm{im}(d_0) \arrow[r] \arrow[d, "\alpha_{X^1}"'] & \mathrm{im}(d_1) \arrow[r] \arrow[d, "\alpha_{X^2}"'] & \cdots \\
\cdots \arrow[r] & \ker(d_0) \arrow[r] \arrow[d, "h_0"'] & \ker(d_1) \arrow[r] \arrow[d, "h_1"'] & \ker(d_1) \arrow[r] \arrow[d, "h_2"'] & \cdots \\
\cdots \arrow[r, "d_{-1}"] & X^0 \arrow[r, "d_0"] & X^1 \arrow[r, "d_1"] & X^2 \arrow[r, "d_2"] & \cdots
\end{tikzcd}$$

By definition, $H^i(X)$ =coker$(\alpha_{X^i}) = \ker(d_i)/\mathrm{im}(\alpha_{X^i})$. In this case, $\mathrm{im}(d_{i-1}) \subseteq \ker(d_i)$ and α_{X^i} is just the inclusion. In particular, α_{X^i} is injective, so $\mathrm{im}(\alpha_{X^i}) \cong \mathrm{im}(d_{i-1})$. Hence, the cokernel is given by $\mathrm{coker}(\alpha_{X^i}) = \ker(d_i)/\mathrm{im}(d_{i-1})$. So $H(X) = \{{}^{\ker(d_i)}/_{\mathrm{im}(d_{i-1})}\}_{i\in\mathbb{Z}}$, and we recover the familiar notion of cohomology!

We would like to construct an additive functor $H : \mathcal{A}_c \longrightarrow \mathcal{A}$ that is given on the objects by cohomology. In that aim, we define $Z(X) = \ker(d_X)$ and $B(X) = \mathrm{im}(T^{-1}(d_X))$. It is a simple verification that this gives rise to a short exact sequence $0 \longrightarrow B(X) \xrightarrow{\alpha_X} Z(X) \xrightarrow{k_X} H(X) \longrightarrow 0$ where the morphism $k_X : Z(X) \to \mathrm{coker}(\alpha_X) = H(X)$ is the epimorphism coming with the definition of the cokernel. Any morphism of differential objects $f : X \to Y$ induces morphisms $B(f) : \mathrm{im}(T^{-1}(d_X)) \longrightarrow \mathrm{im}(T^{-1}(d_Y))$

and $Z(f) : \ker(d_X) \longrightarrow \ker(d_Y)$. Indeed, since f is a morphism of differential objects, the following diagram commutes:

$$\begin{tikzcd}
T^{-1}(X) \arrow[r, "T^{-1}(d_X)"] \arrow[d, "T^{-1}(f)"'] & X \arrow[r, "d_X"] \arrow[d, "f"'] & T(X) \arrow[d, "T(f)"] \\
T^{-1}(Y) \arrow[r, "T^{-1}(d_Y)"'] & Y \arrow[r, "d_Y"'] & T(Y)
\end{tikzcd}$$

from this we deduce that $f(\mathrm{im}(T^{-1}(d_X))) \subseteq \mathrm{im}(T^{-1}(d_Y))$ and $f(\ker(d_X)) \subseteq \ker(d_Y)$.
We get an induced morphism $H(f) : H(X) \longrightarrow H(Y)$ by the universal property of the cokernel:

$$\begin{tikzcd}
H(X) = \mathrm{coker}(\alpha_X) \arrow[dr, dashed, "\exists! H(f)"] & Z(X) \arrow[l, "k_X"'] \arrow[d, "\phi"'] & B(X) \arrow[l, shift right, "\alpha_X"'] \arrow[l, shift left, "0"] \arrow[dl] \\
 & H(Y) = \mathrm{coker}(\alpha_Y) &
\end{tikzcd}$$

Here, $\phi = k_Y \circ Z(f)$ where $k_Y : \ker(d_Y) \longrightarrow \mathrm{coker}(\alpha_Y)$ is given by the universal property of the cokernel. Notice that $\phi \circ \alpha_X = k_Y \circ \alpha_Y \circ B(f) = 0$ by exactness of the short exact sequence given above. It is easy to check that this definition of $H(f)$ turns H into the desired functor and that H is additive.

Example 5.3. We keep the notations from example 5.2. We have seen that in the setting of this example $H(X) = \{\ker(d_i)/\mathrm{im}(d_{i-1})\}_{i\in\mathbb{Z}}$. Consider another chain complex Y with differential denoted by δ and let $\phi : X \longrightarrow Y$ be a chain map. We want to calculate $H(\phi) = \{H(\phi)_i\}_{i\in\mathbb{Z}}$.
Define the quotient maps ${k_X}_i : \ker(d_i) \longrightarrow \ker(d_i)/\mathrm{im}(d_{i-1})$ and ${k_Y}_i : \ker(\delta_i) \longrightarrow \ker(\delta_i)/\mathrm{im}(\delta_{i-1})$. By definition, $H(\phi)_i : \ker(d_i)/\mathrm{im}(d_{i-1}) \longrightarrow \ker(\delta_i)/\mathrm{im}(\delta_{i-1})$ is the unique map such that we have equality between the following compositions $H(\phi)_i \circ {k_X}_i = {k_Y}_i \circ \phi_i|_{\ker(d_i)}$. It is straightforward that the only map satisfying this is defined by $H(\phi)_i([x^i]) = [\phi(x^i)]$, where $x^i \in \ker(d_i)$ and $[x^i]$ denotes its equivalence class in the quotient. The reader might have already encountered this definition of the cohomology functor in algebraic topology. We leave it to him to check that the maps are well-defined and that this is indeed a functor.

That is all nice and good, but it seems like we have drifted away from the subject of triangulated categories. We come back to them in the following proposition, which states that H actually defines a functor from $K_c(\mathcal{A}) \longrightarrow \mathcal{A}$, so it can be seen as a functor from a triangulated category to an abelian category.

Proposition 5.4. *If a morphism of complexes $f : X \to Y$ is homotopic to 0, then the induced morphism $H(f) : H(X) \longrightarrow H(Y)$ is the zero morphism. In particular, H defines a functor $K_c(\mathcal{A}) \longrightarrow \mathcal{A}$.*

Proof. As f is homotopic to 0, there exists $u : X \longrightarrow T^{-1}(Y)$ such that $f = T^{-1}(d_Y) \circ u + T(u) \circ d_X$. Recall that by definition of H we have the following diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & B(X) \arrow[r] \arrow[d, "B(f)"'] & Z(X) \arrow[r, "k_X"] \arrow[d, "Z(f)"'] \arrow[dr] & H(X) \arrow[r] \arrow[d, dashed, "H(f)"] & 0 \\
0 \arrow[r] & B(Y) \arrow[r] & Z(Y) \arrow[r, "k_Y"] & H(Y) \arrow[r] & 0
\end{tikzcd}$$

Since k_X is never zero unless $Z(X)$ is zero, in which case $H(X)$ is zero too, $H(f)$ is zero if and only if the composition $k_Y \circ Z(f) = 0$. Notice that the composition $\ker(d_X) \xrightarrow{u} X \xrightarrow{T(u)\circ d_X} Y$ is the zero morphism. So $Z(f) = Z(T^{-1}(d_Y) \circ u + T(u) \circ d_X) = Z(T^{-1}(d_Y) \circ u)$. Furthermore, $T^{-1}(d_Y) \circ u$ factors through $\mathrm{im}(T^{-1}(d_Y)) = B(Y)$, so by exactness of the bottom row $k_Y \circ Z(T^{-1}(d_Y) \circ u + T(u) \circ d_X) = 0$.
Thus, $H(f) = 0$.
More generally, consider $f, g : X \longrightarrow Y$ two homotopic morphisms. This means $H(f - g) = 0$, but, since H is additive, $0 = H(f - g) = H(f) - H(g)$. Thus, H sends all homotopic morphisms to the same image and therefore defines a functor $K_c(\mathcal{A}) \longrightarrow \mathcal{A}$. $\square$

For a better understanding of what is going on in this proof, let us look at the example of chain complexes again.

Example 5.5. Consider (X,d), (Y,δ) two chain complexes and let $\phi : X \longrightarrow Y$ be a chain map homotopic to 0. Let u be the homotopy, i.e we have that $\phi_i = \delta_{i-1} \circ u_i + u_{i+1} \circ d_i$. The following diagram sums up the situation:

$$\begin{tikzcd}
\cdots \arrow[r, "d_{i-2}"] & X^{i-1} \arrow[r, "d_{i-1}"] \arrow[d, "\phi_{i-1}"'] \arrow[dl, dashed, "u_{i-1}"'] & X^{i} \arrow[r, "d_{i}"] \arrow[d, "\phi_{i}"'] \arrow[dl, dashed, "u_{i}"'] & X^{i+1} \arrow[r, "d_{i+1}"] \arrow[d, "\phi_{i+1}"'] \arrow[dl, dashed, "u_{i+1}"'] & \cdots \arrow[dl, dashed] \\
\cdots \arrow[r, "\delta_{i-2}"'] & Y^{i-1} \arrow[r, "\delta_{i-1}"'] & Y^{i} \arrow[r, "\delta_{i}"'] & Y^{i+1} \arrow[r, "\delta_{i+1}"'] & \cdots
\end{tikzcd}$$

As seen in example 5.3, $H(\phi)_i : \ker(d_i)/\mathrm{im}(d_{i-1}) \longrightarrow \ker(\delta_i)/\mathrm{im}(\delta_{i-1})$ is defined by $H(\phi)_i([x^i]) = [\phi_i(x^i)]$ where $x^i \in \ker(d_i)$. Since ϕ is homotopic to 0, $\phi_i(x^i) = \delta_{i-1} \circ u_i(x^i) + u_{i+1} \circ d_i(x^i)$. By definition, $\delta_{i-1} \circ u_i(x^i) \in \mathrm{im}(\delta_{i-1})$, so this component vanishes in the quotient. In the proof of proposition 5.4, this argument was generalised by showing that $T^{-1}(d_Y) \circ u$ factors through $\mathrm{im}(T^{-1}(d_Y)$ and the argument on exactness. Moreover, $x^i \in \ker(d_i)$, so $u_{i+1} \circ d_i(x^i) = 0$. This was the argument that the composition $\ker(d_X) \xrightarrow{u} X \xrightarrow{T(u)\circ d_X} Y$ is the zero morphism.

Combining both arguments, we see that $[\phi_i(x^i)] = [\delta_{i-1} \circ u_i(x^i) + u_{i+1} \circ d_i(x^i)] = [0]$. The reader might already know this result from algebraic topology, as it is a well-known fact that homotopic spaces have the same cohomology.

As one might know from algebraic topology, cohomology already encodes many informations on the chain complexes. In many settings, it makes sense to look at chain complexes only up to cohomology and to consider to complexes the same if they are isomorphic under cohomology. The following definition formalises this idea.

Definition 5.6. A morphism $f : X \longrightarrow Y$ in $K_c(\mathcal{A})$ is a quasi-isomorphism if $H(f)$ is an isomorphism. The objects X and Y are then called quasi-isomorphic.

Remark 5.7. All isomorphisms are quasi-isomorphisms, since a functor sends isomorphisms to isomorphisms by functoriality. However, the converse is false. As the following example will illustrate, there exist quasi-isomorphisms that are not invertible. Consider the chain map $f : A \to B$ between chain complexes of abelian groups visualised by the following diagram, where B is concentrated at degree 0:

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] & \mathbb{Z} \arrow[r, "\cdot 2"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \arrow[r] & \cdots & A \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0 \arrow[r] & \cdots & B
\end{tikzcd}$$

It is easy to calculate that $H(A) = H(B)$, more precisely $H^i(A) = 0\ \forall i \in \mathbb{Z} \setminus \{0\}$ and $H^0(A) = \mathbb{Z}/2\mathbb{Z}$. It is also easy to verify that $H(f)$ is an isomorphism, in fact $H(f)$ is the identity at every degree. However, there is no way f could be an isomorphism because there exist no non-zero group homomorphism $f^{-1} : \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}$.

As the attentive reader might have suspected when it was announced that H is a functor from a triangulated to an abelian category, H is cohomological. It can thus be a precious tool to understand the triangulated structure. To prove this, we need the following result.

Theorem 5.8. *Let* $0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$ *be a short exact sequence in* $\mathcal{A}_c$. *Then, the sequence* $H(X) \xrightarrow{H(f)} H(Y) \xrightarrow{H(g)} H(Z)$ *is exact and there exist a morphism* $\delta : H(Z) \longrightarrow H(T(X))$ *such that the sequence* $H(X) \xrightarrow{H(f)} H(Y) \xrightarrow{H(g)} H(Z) \xrightarrow{\delta} H(T(X)) \xrightarrow{H(T(f))} H(T(Y))$ *is exact as well.*

The proof requires tools specific to complexes of abelian categories, in particular the snake lemma, which we won't develop here. The proof can be found in [9], theorem 12.2.4.

Corollary 5.9. *H is cohomological. For any distinguished triangle* $X \longrightarrow Y \longrightarrow Z \longrightarrow T(X)$ *in* $K_c(\mathcal{A})$*, the sequence* $H(X) \longrightarrow H(Y) \longrightarrow H(Z)$ *is exact.*

Proof. Consider a distinguished triangle $X \longrightarrow Y \longrightarrow Z \longrightarrow T(X)$. By definition of $K_c(\mathcal{A})$ and TR3, this triangle is isomorphic to $V \overset{\alpha(u)}{\longrightarrow} Mc(u) \overset{\beta(u)}{\longrightarrow} T(U) \longrightarrow T(V)$ for some morphism $u : U \longrightarrow V$. We have seen in definition 4.6 that the sequence $0 \longrightarrow V \overset{\alpha(u)}{\longrightarrow} Mc(u) \overset{\beta(u)}{\longrightarrow} T(U) \longrightarrow 0$ is exact. By theorem 5.8, the sequence $H(V) \xrightarrow{H(\alpha(u))} H(Mc(u)) \xrightarrow{H(\beta(u))} H(T(U))$ is exact. The isomorphism between the triangles yields that $H(X) \longrightarrow H(Y) \longrightarrow H(Z)$ is exact as well. □

Remark 5.10. Many of the results on long exact sequences of cohomolgy used in algebraic topology, for example Mayer-Vietoris, are consequences of theorem 5.8.

Since H is cohomological, it "behaves well" with respect to the triangulated structure. We would like to find a triangulated category that "behaves well" with respect to H in the sense that all quasi-isomorphisms actually are isomorphisms. In particular, for an abelian category of complexes $\mathcal{A}_c$ we would like to construct a category whose objects are the same as those of $\mathcal{A}_c$ and such that two objects are isomorphic if and only if they have isomorphic cohomology. This category will be the derived category. A reader familiar with algebraic geometry might directly think of "localising the homotopy category at quasi-isomorphisms" as one does when one wants to invert certain elements in a ring. But what does it mean to localise a category? Is it even possible? Within the framework of triangulated categories, the most common way to do this is via the so-called Verdier localisation. We will now explain the explicit construction of this localisation, which was first introduced in Verdier's thesis, working in the general setting of arbitrary triangulated categories. Afterwards, we will illustrate how to obtain the derived category, i.e. a category satisfying the properties we just described, as Verdier localisation of the homotopy category.
Until the end of the next section, we closely follow chapter 2 of [16]. Some of the following results being very technical and not very interesting for our understanding of triangulated categories, a few proofs will be left out or only sketched. All detailed proofs can be found in [16]. We also incorporate a few remarks found in [11].

5.2 Verdier localisation

We aim to prove the following theorem due to Verdier:

Theorem 5.11. *Let $\mathcal{D}$ be a triangulated category, $\mathcal{C} \subseteq \mathcal{D}$ a triangulated subcategory (not necessarily thick). Then, there exists a triangulated category denoted by $\mathcal{D}/\mathcal{C}$, and a universal triangulated functor $F_{univ} : \mathcal{D} \longrightarrow \mathcal{D}/\mathcal{C}$ so that $\mathcal{C}$ is in the kernel of F_{univ}, and F_{univ} is universal with this property. If $F : \mathcal{D} \longrightarrow \mathcal{T}$ is a triangulated functor whose kernel contains $\mathcal{C}$, then it factors uniquely as $\mathcal{D} \xrightarrow{F_{univ}} \mathcal{D}/\mathcal{C} \longrightarrow \mathcal{T}$.*

The vocabulary used in this theorem will be defined just a few lines further down. The category $\mathcal{D}/\mathcal{C}$ is called the Verdier quotient of $\mathcal{D}$ by $\mathcal{C}$ and F_{univ} is called the Verdier localisation map. Notice the similarity between this theorem and the universal property of the quotient one might know from group theory. Throughout this section, it will become apparent why this construction is the construction that will allow us to invert all quasi-isomorphisms and finally, in section 5.3, we will construct the derived category of some abelian category $\mathcal{A}$ as Verdier quotient of $K_c(\mathcal{A})$ by a well-chosen subcategory. However, before getting there, we have a lot of beautiful mathematics to do to construct the Verdier localisation and prove theorem 5.11. We proceed step by step. First, we define the notions appearing in the above theorem. Then, we explain how to construct the Verdier quotient. Of course, we detail why everything is well-defined. The third step is to construct the universal functor F_{univ} and show that it is indeed universal. At this point, we will finally have defined all the objects appearing in the theorem. It remains to prove its claim. We begin by showing that the Verdier quotient is additive in step 4. After this, we need to mention a few results on isomorphisms in the quotient. Then, we will at last be in good shape to prove that the Verdier quotient is indeed a triangulated category. So let us begin, as announced, with the definitions.

5.2.1 A Few Definitions

We remind the reader that a subcategory $\mathcal{C}'$ of a category $\mathcal{C}$ is a category such that $Ob(\mathcal{C}') \subseteq Ob(\mathcal{C})$, $Hom_{\mathcal{C}'}(X,Y) \subseteq Hom_{\mathcal{C}}(X,Y)$, the identities in $\mathcal{C}'$ and $\mathcal{C}$ coincide and the composition in $\mathcal{C}'$ is induced by that in $\mathcal{C}$.

Definition 5.12 (full subcategory)**.** A subcategory $\mathcal{C}' \subseteq \mathcal{C}$ is a full subcategory if for all objects of $\mathcal{C}'$ $Hom_{\mathcal{C}'}(X,Y) = Hom_{\mathcal{C}}(X,Y)$.

Definition 5.13 (triangulated subcategory)**.** A subcategory $(\mathcal{C}',T) \subseteq (\mathcal{C},T)$ of a triangulated category $(\mathcal{C},T)$ is a triangulated subcategory if it is a full, additive subcategory, $T(\mathcal{C}) = \mathcal{C}$ and any object in $\mathcal{C}$ isomorphic to some object in $\mathcal{C}'$ is actually in $\mathcal{C}'$. Moreover, we require that if $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$ is a distinguished triangle in $(\mathcal{C},T)$ and $X, Y \in (\mathcal{C}',T)$, then $Z \in (\mathcal{C}',T)$ as well. This condition guarantees that for every morphism in $(\mathcal{C}',T)$ the triangle given by TR2 extending it is in $(\mathcal{C}',T)$.

It follows from the definition that a triangulated subcategory is triangulated with the triangulation being the one of $(\mathcal{C},T)$ restricted to $(\mathcal{C}',T)$. In other words, triangles in $(\mathcal{C}',T)$ are those that are already distinguished when seen as triangles in $(\mathcal{C},T)$.

Definition 5.14 (thick subcategory)**.** A subcategory of a triangulated category is called thick if it is a triangulated subcategory that contains all direct summands of its objects.

Definition 5.15 (triangulated functor)**.** A functor $F : (\mathcal{D}_1,T_1) \longrightarrow (\mathcal{D}_2,T_2)$ between two triangulated categories is said to be triangulated if it is a functor of additive categories and for every object $X \in \mathcal{D}_1$ there exists a natural isomorphism $\phi_X : F(T_1(X)) \longrightarrow T_2(F(X))$ such that for any distinguished triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T_1(X)$ in $\mathcal{D}_1$, the triangle $F(X) \xrightarrow{F(u)} F(Y) \xrightarrow{F(v)} F(Z) \xrightarrow{\phi_X \circ F(w)} T_2(F(X))$ is distinguished in $\mathcal{D}_2$.
This condition allows us to drop the distinction between T_1 and T_2 and we will often just write T for any of the two.

One can observe that these definitions make the inclusion functor from a triangulated subcategory into a triangulated category a triangulated functor and in this setting ϕ_X is the identity on $T(X)$.

Definition 5.16 (Kernel of a triangulated functor)**.** Let $F : (\mathcal{D}_1,T) \longrightarrow (\mathcal{D}_2,T)$ be a triangulated functor. The kernel of F, denoted $\mathrm{Ker}(F)$, is the full subcategory of $\mathcal{D}_1$ whose objects are such that their image by F is isomorphic to 0 i.e $Ob(\mathrm{Ker}(F)) = \{X \in Ob(\mathcal{D}_1) \mid F(X) \cong 0\}$.

Proposition 5.17. *In the setting as in the defintion, Ker(F) is a thick subcategory of* $(\mathcal{D}_1,T)$.

Proof. If $X \cong Y$ in $(\mathcal{D}_1,T)$ and $F(X) \cong 0$, then $F(Y) \cong 0$ too, since $F(X) \cong F(Y)$ by functoriality. So it is clear that any object isomorphic to an object in the kernel is in the kernel.
Next observe that, for X an object in $(\mathcal{D}_1,T)$, $F(X) \cong 0$ if and only if $T(F(X)) \cong F(T(X)) \cong 0$. This implies that $T(X) \in \mathrm{Ker}(F)$ if and only if $X \in \mathrm{Ker}(F)$. Let $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$ be a distinguished triangle in $(\mathcal{D}_1,T)$ with $X, Y \in \mathrm{Ker}(F)$. We want to show that $Z \in \mathrm{Ker}(F)$ as well. Since F is a triangulated functor, $F(X) \xrightarrow{F(u)} F(Y) \xrightarrow{F(v)} F(Z) \xrightarrow{\phi_X \circ F(w)} T(F(X))$ is a distinguished triangle in $(\mathcal{D}_2,T)$. But $F(X) \cong F(Y) \cong T(F(X)) \cong 0$. So:

$$\begin{tikzcd}
F(X) \arrow[r, "F(u)"] \arrow[d, "\sim"'] & F(Y) \arrow[r, "F(v)"] \arrow[d, "\sim"'] & F(Z) \arrow[r, "\phi_X \circ F(w)"] \arrow[d, dashed] & T(F(X)) \arrow[d, "\sim"] \\
0 \arrow[r, equal] & 0 \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

The bottom line is distinguished by TR0, the dashed arrow exists by TR4 and is an isomorphism by proposition 3.22. Thus, $F(Z) \cong 0$ i.e. $Z \in \mathrm{Ker}(F)$. Thus, $\mathrm{Ker}(F)$ is a triangulated subcategory.
Furthermore, we need to check that if $X \oplus Y \in \mathrm{Ker}(F)$, then $X, Y \in \mathrm{Ker}(F)$. Since F is additive, $F(X \oplus Y) \cong F(X) \oplus F(Y)$. By assumption, $F(X \oplus Y) \cong 0$, so $F(X)$ and $F(Y)$ are direct summands of 0 thus isomorphic to 0. $\square$

5.2.2 Defining the Verdier quotient

Having now defined the terms appearing in the theorem, we focus on constructing the Verdier quotient. On the objects, $\mathcal{D}/\mathcal{C}$ will be defined in the same way as $\mathcal{D}$ i.e. $Ob(\mathcal{D}/\mathcal{C}) = Ob(\mathcal{D})$. In order to define the morphisms, we need to make a few more observations. Until the end of this chapter, $\mathcal{D}$ will always denote a triangulated category and $\mathcal{C}$ will always be a triangulated subcategory of $\mathcal{D}$.

Definition 5.18. We define a collection of morphisms $Mor_{\mathcal{C}} \subseteq D$ in the following manner:
A morphism $f : X \to Y$ of $\mathcal{D}$ is in $Mor_{\mathcal{C}}$ if and only if for some distinguished triangle $X \xrightarrow{f} Y \to Z \to T(X)$ extending f, the object Z is in $\mathcal{C}$.

It doesn't matter which triangle we choose as we have seen in corollary 3.24 that all triangles extending a given morphism are isomorphic triangles and, by definition of triangulated subcategory, $\mathcal{C}$ is closed with respect to isomorphic objects.

Observe that for any triangulated functor F such that $\mathcal{C} \subseteq \mathrm{Ker}(F)$ and for any morphism $f : X \longrightarrow Y$ in $Mor_{\mathcal{C}}$, the morphism $F(f) : F(X) \longrightarrow F(Y)$ is an isomorphism. Indeed, by definition of the kernel, any object $X \in \mathcal{C}$ will satisfy $F(X) \cong 0$. Since f is in $Mor_{\mathcal{C}}$, the distinguished triangle extending it $X \xrightarrow{f} Y \longrightarrow Z \longrightarrow T(X)$ is such that $Z \in \mathcal{C}$. As F is a triangulated functor, the triangle $F(X) \xrightarrow{F(f)} F(Y) \longrightarrow F(Z) \longrightarrow T(F(X))$ is distinguished. But $F(Z) \cong 0$, so, by proposition 3.25, $F(f)$ is an isomorphism. In other words, all morphisms in $Mor_{\mathcal{C}}$ are invertible in $\mathcal{D}/\mathcal{C}$ (since $\mathcal{C}$ will be in the kernel of the triangulated functor F_{univ}).
Remember that we had introduced the Verdier localisation with the promise that, in the quotient of the homotopy category with an adequate subcategory, all quasi-isomorphisms will turn into isomorphisms. This observation hints that we shall choose the subcategory $\mathcal{C}$ such that $Mor_{\mathcal{C}}$ contains all quasi-isomorphism. Moreover, this observation motivates the idea to construct the morphisms of the quotient as composition of morphisms in $\mathcal{D}$ and formal inverses of the morphisms in $\mathcal{C}$. To make this idea rigorous, we need to introduce a notion that will be useful when defining the composition of morphisms in the quotient and state a few properties of $Mor_{\mathcal{C}}$.

Definition 5.19 (homotopy cartesian square)**.** A homotopy cartesian square is a commutative square in $\mathcal{D}$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "g'"] \\ X' \arrow[r, "f'"'] & Y' \end{tikzcd}$$

for which there exists a morphism $d : Y' \to T(X)$ such that the triangle $X \xrightarrow{\binom{g}{-f}} X' \oplus Y \xrightarrow{(f'\ g')} Y' \xrightarrow{d} T(X)$ is distinguished. The object X in the upper-left of the square is called the homotopy pullback of

$$\begin{tikzcd} & Y \arrow[d, "g'"] \\ X' \arrow[r, "f'"'] & Y' \end{tikzcd}$$

and the object Y' in the lower-right of the square is called the homotopy pushout of

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \\ X' & \end{tikzcd}$$

Remark 5.20. In a triangulated category, homotopy pushouts and pullbacks always exist. Indeed, by TR2 the morphism $\binom{g}{-f} : X \longrightarrow X' \oplus Y$ can be completed to a distinguished triangle

$$X \xrightarrow{\binom{g}{-f}} X' \oplus Y \longrightarrow Y' \longrightarrow T(X)$$

for some Y'. This triangle defines a homotopy cartesian square. The existence of the pull-back is shown similarly by extending the morphism $\begin{pmatrix} f' & g' \end{pmatrix} : X' \oplus Y \longrightarrow Y'$ to a distinguished triangle and applying TR3. Recall from corollary 3.24 that the triangle extending a given morphism is unique up to isomorphism. From the above construction, this implies homotopy pullouts and pushbacks are unique up to isomorphism as well.

We briefly state a few nice and useful properties of $Mor_{\mathcal{C}}$. Elaborate proofs can be found in [16], lemma 1.5.6 - 1.5.8.

Proposition 5.21.
1. *Every isomorphism in $f : X \longrightarrow Y$ is in $Mor_{\mathcal{C}}$. This is due to the fact that since $\mathcal{C}$ is additive, $0 \in \mathcal{C}$ and triangles extending isomorphisms factor through 0 (prop 3.25).*
2. *As a consequence of TR5, if any two of the following $f : X \longrightarrow Y$, $g : Y \longrightarrow Z$ or $g \circ f : X \longrightarrow Z$ are in $Mor_{\mathcal{C}}$, then so is the third.*
3. *In particular, one can compose morphisms in $Mor_{\mathcal{C}}$ and together with all objects of $\mathcal{D}$ it defines a category.*
4. *$Mor_{\mathcal{C}}$ contains all homotopy pushouts and pullbacks of its morphisms. In other words, for a homotopy cartesian square*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "g'"] \\ X' \arrow[r, "f'"'] & Y' \end{tikzcd}$$

$f \in Mor_{\mathcal{C}}$ if and only if f' is and $g \in Mor_{\mathcal{C}}$ if and only if g' is.

These properties become intuitive, if, as motivated before, we think of $Mor_{\mathcal{C}}$ as the collection of quasi-isomorphisms. Then, property 1) states the obvious fact that isomorphisms are quasi-isomorphisms and the other properties assure that quasi-isomorphisms behave well with respect to composition.
We now are a step closer to defining the morphisms in the quotient. We do this by formally inverting the morphisms in $Mor_{\mathcal{C}}$ and identifying equivalent compositions in a similar way as one identifies the fractions $\frac{1}{2}$ and $\frac{2}{4}$. This is formalised in the next definition.

Definition 5.22. Let $X, Y \in Ob(\mathcal{D})$. We define $\alpha(X, Y)$ as the class of diagrams of the form

$$X \xleftarrow{f} Z \xrightarrow{g} Y$$

with $f \in Mor_{\mathcal{C}}$. We will write (Z, f, g) to denote such a diagram.
We define the relation $R(X, Y)$ on $\alpha(X, Y)$ such that two diagrams $(Z, f, g), (Z', f', g')$ belong to $R(X, Y)$ if and only if there exists a diagram $(Z'', f'', g'') \in \alpha(X, Y)$ and morphisms $u : Z'' \longrightarrow Z$, $v : Z'' \longrightarrow Z'$ such that the following diagram commutes.

$$\begin{tikzcd} & Z' \arrow[dl, "f'"'] \arrow[dr, "g'"] & \\ X & Z'' \arrow[l, "f''"'] \arrow[r, "g''"] \arrow[u, "v"'] \arrow[d, "u"] & Y \\ & Z \arrow[ul, "f"] \arrow[ur, "g"'] & \end{tikzcd}$$

One can easily check that $R(X, Y)$ is an equivalence relation. The proof can be found in [16], lemma 2.1.14.

Proposition 5.23. *In the above setting, if u and v exist, they are in $Mor_{\mathcal{C}}$.*

Proof. By definition, $f \circ u = f''$ and $f, f'' \in Mor_{\mathcal{C}}$. By proposition 5.21, also $u \in Mor_{\mathcal{C}}$, The claim for v follows analogously. $\square$

Remember that we want any morphism in $Mor_{\mathcal{C}}$ to be invertible in $\mathcal{D}/\mathcal{C}$. One should think of the maps in $\alpha(X, Y)$ as morphisms $g \circ f^{-1} \in \mathcal{D}/\mathcal{C}$. Then, $R(X, Y)$ identifies equal morphisms, as, if the relation is satisfied,

$$g \circ f^{-1} = g \circ u \circ u^{-1} \circ f^{-1} = g'' \circ (f'')^{-1} = g' \circ v \circ v^{-1} \circ f'^{-1} = g' \circ f'^{-1}.$$

We can finally define the morphisms of ${}^{\mathcal{D}}\!/_{\mathcal{C}}$:

Definition 5.24. $Hom_{\mathcal{D}/\mathcal{C}}(X,Y)$ is the class of equivalence classes in $\alpha(X,Y)$ with respect to $R(X,Y)$.

We still need to define the identities and the composition and check the latter is associative before we know that the Verdier quotient defined above is really a category.
Consider an element $(W_1, f_1, g_1) \in \alpha(X,Y)$ and an element $(W_2, f_2, g_2) \in \alpha(Y,Z)$. How can we compose these diagrams? The most natural idea is to find a diagram $(W_3, \phi, \psi) \in \alpha(W_1, W_2)$ completing

$$\begin{tikzcd}
W_3 \arrow[r, dashed, "\psi"] \arrow[d, dotted, "\phi"'] & W_2 \arrow[r, "g_2"'] \arrow[d, "f_2"] & Z \\
W_1 \arrow[r, "g_1"'] \arrow[d, "f_1"'] & Y & \\
X & &
\end{tikzcd}$$

This construction works, (W_3, ϕ, ψ) is obtained by homotopy pullback of

$$\begin{tikzcd}
 & W_2 \arrow[d, "f_2"] \\
W_1 \arrow[r, "g_1"'] & Y
\end{tikzcd}$$

Remember that by remark 5.21 4), $\phi \in Mor_{\mathcal{C}}$ because f_2 is in $Mor_{\mathcal{C}}$. Moreover, since $f_1 \in Mor_{\mathcal{C}}$, the composition $\phi \circ f_1$ is in $Mor_{\mathcal{C}}$ as well. Thus $(W_3, \phi \circ f_1, g_2 \circ \psi) \in \alpha(X,Z)$. Taking the equivalence class, this construction yields a composition map

$$\begin{aligned} \alpha(X,Y) \times \alpha(Y,Z) &\longrightarrow Hom_{\mathcal{D}/\mathcal{C}}(X,Z) \\ ((W_1,f_1,g_1),(W_2,f_2,g_2)) &\mapsto [(W_3, \phi \circ f_1, g_2 \circ \psi)] \end{aligned} \tag{21}$$

As the following proposition asserts, this turns $\mathcal{D}/\mathcal{C}$ into a well-defined category.

Proposition 5.25. *The map* (21) *is consistent with the equivalence relations $R(X,Y)$ and $R(Y,Z)$. It therefore induces a map $Hom_{\mathcal{D}/\mathcal{C}}(X,Y) \times Hom_{\mathcal{D}/\mathcal{C}}(Y,Z) \longrightarrow Hom_{\mathcal{D}/\mathcal{C}}(X,Z)$. Moreover, this map is associative and (X, id_X, id_X) is a two-sided identity element for this composition. In particular, these definitions make $\mathcal{D}/\mathcal{C}$ a category.*

The proof consists of straightforward, but lengthy manipulations of the diagrams. It is left to the reader, but can also be found in [16], lemma 2.1.18 and 2.1.19.
We at last have a rigorous definition of the Verdier quotient, but we still have little understanding of how to concretely work with it. The following three propositions give nicer descriptions of the composition and inverse of morphisms.

Proposition 5.26. *Let $(W_1, f_1, g_1) \in \alpha(X,Y)$ and $(W_2, f_2, g_2) \in \alpha(Y,Z)$. Suppose there exists a commutative diagram with $\phi' \in Mor_{\mathcal{C}}$:*

$$\begin{tikzcd}
P \arrow[r, "\psi'"] \arrow[d, "\phi'"'] & W_2 \arrow[r, "g_2"] \arrow[d, "f_2"'] & Z \\
W_1 \arrow[r, "g_1"'] \arrow[d, "f_1"'] & Y & \\
X & &
\end{tikzcd}$$

Then $(P, \phi' \circ f_1, g_2 \circ \psi')$ agrees in $Hom_{\mathcal{D}/\mathcal{C}}(X,Z)$ with $(W_3, \phi \circ f_1, g_2 \circ \psi)$ obtained as homotopy pullback. In other words, one can choose any diagram in $\alpha(W_1, W_2)$ that makes the square commute to define the composition. One doesn't need to check that it is a homotopy pullback.

Proof. We wish to construct a morphism $\alpha : P \longrightarrow W_3$ such that $\phi \circ \alpha = \phi'$ and $\psi \circ \alpha = \psi'$. If we had such a map, $(P, \phi' \circ f_1, g_2 \circ \psi')$ and $(W_3, \phi \circ f_1, g_2 \circ \psi)$ would be equivalent in $Hom_{\mathcal{D}/\mathcal{C}}(X, Z)$ because

$$\begin{tikzcd}
 & & W_3 \arrow[dl, "\phi"'] \arrow[dr, "\psi"] & & \\
X & W_1 \arrow[l, "f_1"'] & P \arrow[l, dashed, "\phi'"'] \arrow[r, dashed, "\psi'"] \arrow[u, dashed, "\alpha"] \arrow[d] & W_2 \arrow[r, "g_2"] & Z \\
 & & P \arrow[ul, "\phi'"] \arrow[ur, "\psi'"'] & &
\end{tikzcd}$$

To construct α, notice that, by hypothesis, $g \circ \phi' = f_1 \circ \psi'$ and thus $\begin{pmatrix} f_1 & g_2 \end{pmatrix} \circ \begin{pmatrix} \psi' \\ -\phi' \end{pmatrix} = 0$. Since W_3 is a homotopy pullback, the triangle $W_3 \xrightarrow{\begin{pmatrix} \psi \\ -\phi \end{pmatrix}} W_1 \oplus W_2 \xrightarrow{\begin{pmatrix} f_1 & g_2 \end{pmatrix}} Y \longrightarrow T(W_3)$ is distinguished. Remember from proposition 3.16 that $Hom(P, -)$ is cohomological. Hence, we get an exact sequence

$$Hom(P, W_3) \xrightarrow{\begin{pmatrix} \psi \\ -\phi \end{pmatrix}_\star} Hom(P, W_1 \oplus W_2) \xrightarrow{\begin{pmatrix} f_1 & g_2 \end{pmatrix}_\star} Hom(P, Y)$$

By the previous observation, $\begin{pmatrix} f_1 & g_2 \end{pmatrix}_\star \left(\begin{pmatrix} \psi' \\ -\phi' \end{pmatrix}\right) = 0$. By exactness, $\begin{pmatrix} \psi' \\ -\phi' \end{pmatrix} \in \operatorname{im}\left(\begin{pmatrix} \psi \\ -\phi \end{pmatrix}_\star\right)$, so there exists a map $\alpha \in Hom(P, W_3)$ such that $\begin{pmatrix} \psi \circ \alpha \\ -\phi \circ \alpha \end{pmatrix} = \begin{pmatrix} \psi' \\ -\phi' \end{pmatrix}$. This is the map we were looking for. $\square$

Recall we constructed the quotient by formally inverting the morphisms of $Mor_{\mathcal{C}}$. The following proposition states that we succeeded, that all morphisms of $Mor_{\mathcal{C}}$ are isomorphisms in the quotient. In particular, if we manage to construct the derived category as Verdier quotient of the homotopy category and a subcategory $\mathcal{C}$ such that $Mor_{\mathcal{C}}$ consists of all quasi-isomorphisms, by the following proposition, we have reached our goal of inverting all quasi-isomorphisms in the derived category.

Proposition 5.27. *Let $f : X \longrightarrow Y$ be a morphism in $Mor_{\mathcal{C}}$. In $\mathcal{D}/\mathcal{C}$, $X =\!= X \xrightarrow{f} Y$ and $Y \xleftarrow{f} X =\!= X$ are inverse to each other.*

Proof. We have the following commutative diagram:

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r, equal] \arrow[d, "f"] & X \\
X \arrow[r, "f"] \arrow[d, equal] & Y & \\
X & &
\end{tikzcd}$$

In view of proposition 5.26, we deduce that $[(X, f, id_X)] \circ [(X, id_X, f)] = [(X, id_X, id_X)]$.
From this diagram

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r, "f"] \arrow[d, equal] & Y \\
X \arrow[r, equal] \arrow[d, "f"'] & X & \\
Y & &
\end{tikzcd}$$

we see that $[(X, id_X, f)] \circ [(X, f, id_X)] = [(X, f, f)]$. But using that $f \in Mor_{\mathcal{C}}$, we conclude from the following diagram that $[(X, f, f)] = [(Y, id_Y, id_Y)]$.

$$\begin{tikzcd}
 & Y \arrow[dl, equal] \arrow[dr, equal] & \\
Y & X \arrow[l, dashed, "f"'] \arrow[r, dashed, "f"] \arrow[u, dashed, "f"] \arrow[d, equal] & Y \\
 & X \arrow[ul, "f"] \arrow[ur, "f"'] &
\end{tikzcd}$$

$\square$

Finally, the next proposition gives us a deeper understanding of what the morphisms in the quotient look like.

Proposition 5.28. *One can write any morphism* $X \xleftarrow{f} W \xrightarrow{g} Y$ *in* ${}^{\mathcal{D}}\!/_{\mathcal{C}}$ *as composition of* $X \xleftarrow{f} W = W$ *and* $W = W \xrightarrow{g} Y$ *.*

Proof. The following diagram combined with proposition 5.26 proves the claim.

$$\begin{tikzcd} W \arrow[r, equal] \arrow[d, equal] & W \arrow[r, "g"] \arrow[d, equal] & Y \\ W \arrow[r, equal] \arrow[d, "f"'] & W & \\ X & & \end{tikzcd}$$

□

5.2.3 Defining the universal functor

In order to prove theorem 5.11, we still need to show that $\mathcal{D}/\mathcal{C}$ is triangulated, in particular we need to prove it is additive and define a translation on it. Moreover, we need to define F_{univ}, prove that it is a triangulated functor and universal. Let us begin with the latter. Recall that the categories $\mathcal{D}$ and $\mathcal{D}/\mathcal{C}$ coincide on the objects and that by proposition 5.28 the morphism of the quotient are of the form "$g \circ f^{-1}$" with $g \in \mathcal{D}$ and $f \in Mor_{\mathcal{C}}$. It thus seems natural to think of F_{univ} as an "inclusion" of $\mathcal{D}$ into the quotient. The following definition makes this idea rigorous.

Definition 5.29. The functor $F_{univ} : \mathcal{D} \longrightarrow {}^{\mathcal{D}}\!/_{\mathcal{C}}$ is the identity on the objects and for a morphism $f : X \longrightarrow Y$ in $\mathcal{D}$, $F(f)$ is the equivalence class of the diagram $X = X \xrightarrow{f} Y$.

Proposition 5.30. *F_{univ} is really a functor. Moreover, if $f \in Mor_{\mathcal{C}}$, $F_{univ}(f)$ is invertible and any morphism of $\mathcal{D}/\mathcal{C}$ can be written as $F_{univ}(g) \circ F_{univ}(f)^{-1}$ for some $g \in \mathcal{D}$ and $f \in Mor_{\mathcal{C}}$.*

Proof. We need to check that for $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$, $F_{univ}(g \circ f) = F_{univ}(g) \circ F_{univ}(f)$ and that $F_{univ}(id_X) = id_{F_{univ}(X)}$. The latter is clear since $F_{univ}(id_X)$ is given by $X = X = X$ which is the identity of X in the quotient. The composition $F_{univ}(g \circ f) = X = X \xrightarrow{g\circ f} Z$, $F_{univ}(f)$ is given by $X = X \xrightarrow{f} Y$ and $F_{univ}(g)$ is $Y = Y \xrightarrow{g} Z$. The composition $F_{univ}(g) \circ F_{univ}(f)$ is given by:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "g"] \arrow[d, equal] & Z \\ X \arrow[r, "f"] \arrow[d, equal] & Y & \\ X & & \end{tikzcd}$$

which proves the equality. Thus, F_{univ} is a functor.

The fact that $F_{univ}(f)$ is invertible for any $f \in Mor_{\mathcal{C}}$ follows directly from the definition of F_{univ} and proposition 5.27 and the fact that any morphism can be decomposed is a direct consequence of proposition 5.28.

□

We say that a morphism of $\mathcal{D}$ is an isomorphism in the quotient if $F(f)$ is an isomorphism. In particular, we have just seen that all morphisms of $Mor_{\mathcal{C}}$ are isomorphisms in the quotient. Since we would like to think of F_{univ} as an inclusion, it is natural to wonder if two different morphisms of $\mathcal{D}$ can have the same F_{univ}-image. As claimed by the next proposition, it turns out that this can happen, but under conditions that are weak enough for us to still visualise F_{univ} as an "inclusion".

Proposition 5.31. *Let $f, g : X \longrightarrow Y$ be two morphisms in $\mathcal{D}$. Then, $F_{univ}(f) = F_{univ}(g)$ if and only if there exists a morphism $\alpha : W \longrightarrow X \in Mor_{\mathcal{C}}$ such that $f \circ \alpha = g \circ \alpha$.*

Proof. $F_{univ}(f) = F_{univ}(g)$ if and only if the diagrams $X = X \xrightarrow{f} Y$ and $X = X \xrightarrow{g} Y$ are equivalent i.e. there exists an object $W \in \mathcal{D}$ and maps $\alpha_1 : W \longrightarrow X$, $\alpha_2 : W \longrightarrow X$ in $Mor_\mathcal{C}$ such that

$$\begin{tikzcd}
& X \arrow[dr, "f"] & \\
X \arrow[ur, equal] \arrow[dr, equal] & W \arrow[l] \arrow[u, dashed, "\alpha_1"] \arrow[d, dashed, "\alpha_2"'] \arrow[r] & Y \\
& X \arrow[ur, "g"'] &
\end{tikzcd}$$

The commutativity of the left half implies $\alpha_1 = \alpha_2 := \alpha$, the right half gives $f \circ \alpha = g \circ \alpha$. $\square$

The functor F_{univ} is very useful because it allows to lift statement from $\mathcal{D}$ to the quotient. This will be crucial when proving that the Verdier quotient of a triangulated category is triangulated. The following proposition is an example of such an application of F_{univ} that we will need in the proof of theorem 5.11.

Proposition 5.32. *Any commutative square in the quotient is isomorphic to the image by F_{univ} of a commutative square in $\mathcal{D}$.*

In other words, if $\begin{array}{ccc} W & \xrightarrow{a} & X \\ \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle b} \\ Y & \xrightarrow{d} & Z \end{array}$ *is a commutative square in the quotient, then there exists a commutative square in $\mathcal{D}$* $\begin{array}{ccc} W' & \xrightarrow{\alpha} & X' \\ \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \beta} \\ Y' & \xrightarrow{\delta} & Z \end{array}$ *and maps $\xi : W' \longrightarrow W$, $f_2 : X' \longrightarrow X$ and $f_4 : Y' \longrightarrow Y$ in $Mor_\mathcal{C}$ such that the following diagram commutes:*

$$\begin{tikzcd}
& F_{univ}(W) = W \arrow[rr, "a"] \arrow[dd, "c"] & & F_{univ}(X) = X \arrow[dd, "b"] \\
F_{univ}(W') \arrow[ur, "F_{univ}(\xi)"] \arrow[rr, "F_{univ}(\alpha)"] \arrow[dd, "F_{univ}(\gamma)"'] & & F_{univ}(X') \arrow[ur, "F_{univ}(f_2)"] \arrow[dd, "F_{univ}(\beta)"] & \\
& F_{univ}(Y) = Y \arrow[rr, "d"] & & F_{univ}(Z) = Z \\
F_{univ}(Y') \arrow[ur, "F_{univ}(f_4)"] \arrow[rr, "F_{univ}(\delta)"'] & & F_{univ}(Z) \arrow[ur, equal] &
\end{tikzcd}$$

Since $f_2, f_4, \xi \in Mor_\mathcal{C}$, $F_{univ}(f_2), F_{univ}(f_4), F_{univ}(\xi)$ are isomorphisms.

Proof. We construct the square in $\mathcal{D}$ and the isomorphisms explicitly. The composition $b \circ a$ is given by:

$$\begin{tikzcd}
W_1 \arrow[r, dashed, "\psi"] \arrow[d, dashed, "\phi"'] & X' \arrow[r, "g_2"] \arrow[d, "f_2"'] & Z \\
\tilde{W}_1 \arrow[r, "g_1"] \arrow[d, "f_1"'] & X \arrow[ur, "b"'] & \\
W \arrow[ur, "a"'] & &
\end{tikzcd}$$

with all vertical arrows in $Mor_\mathcal{C}$. Similarly, the composition $d \circ c$ can be visualised as:

$$\begin{tikzcd}
W_2 \arrow[r, dashed, "\psi'"] \arrow[d, dashed, "\phi'"'] & Y' \arrow[r, "g_4"] \arrow[d, "f_4"'] & Z \\
\tilde{W}_2 \arrow[r, "g_3"] \arrow[d, "f_3"'] & Y \arrow[ur, "d"'] & \\
W \arrow[ur, "c"'] & &
\end{tikzcd}$$

Consider W_3 the homotopy pullback of W_2 and W_1 as drawn in the diagram: $\begin{tikzcd} W_3 \arrow[r, "\eta"] \arrow[d, "\theta"'] & W_2 \arrow[d, "f_3\circ\phi'"] \\ W_1 \arrow[r, "f_1\circ\phi"] & W \end{tikzcd}$

Recall that by proposition 5.21, $\eta, \theta \in Mor_{\mathcal{C}}$ because $f_3 \circ \phi', f_1 \circ \phi \in Mor_{\mathcal{C}}$. The square $\begin{tikzcd} W_3 \arrow[r] \arrow[d] & X' \arrow[d] \\ Y' \arrow[r] & Z \end{tikzcd}$ is isomorphic in the quotient (applying F_{univ}) to $\begin{tikzcd} W \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd}$, thus it commutes in the quotient. The isomorphism is given by:

$$\begin{tikzcd}
W_3 \arrow[r, "\eta"] \arrow[d, "\theta"] & W_1 \arrow[rr] \arrow[d, dashed, "f_1\circ\phi"] & & X' \arrow[dl, dashed, "f_2"] \arrow[ddd] \\
W_2 \arrow[r, dashed, "f_3\circ\phi'"] \arrow[dd] & W \arrow[r] \arrow[d] & X \arrow[d] & \\
& Y \arrow[r] & Z \arrow[dr, equal] & \\
Y' \arrow[ur, dashed, "f_4"] \arrow[rrr] & & & Z
\end{tikzcd}$$

where all the dashed arrows are in $Mor_{\mathcal{C}}$, so they are isomorphisms in the quotient.
However, even though the outer square commutes in the quotient, it might not yet commute in $\mathcal{D}$. But, by proposition 5.31, commutativity in the quotient implies there must exist an object $W' \in \mathcal{D}$ and a morphism $\varphi : W' \longrightarrow W_3$ in $Mor_{\mathcal{C}}$ such that precomposition with α equalises the compositions in $\mathcal{D}$. This gives the wanted square. The situation is summed up by the following diagram:

$$\begin{tikzcd}
W' \arrow[r] \arrow[d] \arrow[dr, dashed, "\varphi"] & W_1 \arrow[rrr] \arrow[ddr, dashed] & & & X' \arrow[ddl, dashed] \arrow[dddd] \\
W_2 \arrow[drr, dashed] \arrow[ddd] & W_3 \arrow[l] \arrow[u] & & & \\
& & W \arrow[r] \arrow[d] & X \arrow[d] & \\
& & Y \arrow[r] & Z \arrow[dr, equal] & \\
Y' \arrow[urr, dashed] \arrow[rrrr] & & & & Z
\end{tikzcd}$$

$\square$

We end the subsection on the universal functor by checking that F_{univ} as defined here really is a universal functor in the sense of theorem 5.11. In doing so, we already prove a tiny fraction of this theorem.

Proposition 5.33. *F_{univ} is universal, meaning if $F : \mathcal{D} \longrightarrow \mathcal{T}$ is a functor that takes morphisms in $Mor_{\mathcal{C}}$ to invertible morphisms, then it factors uniquely as* $\mathcal{D} \xrightarrow{F_{univ}} \mathcal{D}/\mathcal{C} \longrightarrow \mathcal{T}$.

Proof. We define a functor $\tilde{F} : \mathcal{D}/\mathcal{C} \longrightarrow \mathcal{T}$. On the objects $\tilde{F} = F$. On the morphisms, $\tilde{F}$ sends the diagram $X \xleftarrow{f} W_1 \xrightarrow{g} Y$ to $F(g) \circ F(f)^{-1}$. $F(f)^{-1}$ is well-defined by hypothesis. With this definition, it is clear that $F = \tilde{F} \circ F_{univ}$. We need to check that $\tilde{F}$ is well-defined in the sense that it sends diagrams that are equivalent modulo $R(X, Y)$ to the same morphism.
Suppose (W_2, f', g') and (W_1, f, g) are two equivalent diagrams in $\alpha(X, Y)$. We get the existence of

(W_3, f'', g'') such that the following diagram commutes:

$$\begin{tikzcd} & W_2 \arrow[dl, "f'"'] \arrow[dr, "g'"] & \\ X & W_3 \arrow[l, "f''"'] \arrow[r, "g''"] \arrow[u, "v"'] \arrow[d, "u"] & Y \\ & W_1 \arrow[ul, "f"] \arrow[ur, "g"'] & \end{tikzcd}$$

Recall from proposition 5.23, that $u, v \in Mor_{\mathcal{C}}$. Using this and functoriality of F, we prove that $\tilde{F}$ is well-defined:

$$\tilde{F}((W_1, f, g)) = F(g) \circ F(f)^{-1} = F(g) \circ F(u) \circ F(u)^{-1} \circ F(f)^{-1} =$$
$$F(g \circ u) \circ F(f \circ u)^{-1} = F(g'') \circ F(f'')^{-1} = F(g' \circ v) \circ F(f' \circ v)^{-1} =$$
$$F(g') \circ F(v) \circ F(v)^{-1} \circ F(f')^{-1} = F(g') \circ F(f')^{-1} = \tilde{F}((W_2, f', g'))$$

Moreover, we check that $\tilde{F}$ is a functor. It clearly sends identities to identities and for two equivalence classes of diagrams $[(W_1, f_1, g_1)]$ and $[(W_2, f_2, g_2)]$ in $\alpha(X, Y)$ and $\alpha(Y, Z)$ respectively:

$$\tilde{F}([(W_2, f_2, g_2)] \circ [(W_1, f_1, g_1)]) = \tilde{F}([(W_3, f_1 \circ \phi, g_1 \circ \psi)]) =$$
$$F(g_2 \circ \psi) \circ F(f_2 \circ \psi)^{-1} \circ F(g_1 \circ \phi) \circ F(f_1 \circ \phi)^{-1} =$$
$$F(g_2) \circ F(\psi) \circ F(\psi)^{-1} \circ F(f_2)^{-1} \circ F(g_1) \circ F(\phi) \circ F(\phi)^{-1} \circ F(f_1)^{-1} =$$
$$F(g_2) \circ F(f_2)^{-1} \circ F(g_1) \circ F(f_1)^{-1} = \tilde{F}([(W_2, f_2, g_2)]) \circ \tilde{F}([(W_1, f_1, g_1)])$$

Finally, we show that $\tilde{F}$ is the only functor satisfying $F = \tilde{F} \circ F_{univ}$.
First of all, it is clear that any functor F' satisfying $F = F' \circ F_{univ}$ must be defined in the same way as F on the objects. So F' and $\tilde{F}$ must coincide on the objects of the quotient.
Remember from proposition 5.28, that any morphism in the quotient $X \xleftarrow{f} W_1 \xrightarrow{g} Y$ can be written as composition $X \xleftarrow{f} W_1 = W_1$ and $W_1 = W_1 \xrightarrow{g} Y$. By functoriality, it is therefore sufficient to show that F' and $\tilde{F}$ coincide on the above diagrams. The second diagram is precisely $F_{univ}(g)$. Since $F = F' \circ F_{univ}$, we must have $F'(\ W_1 = W_1 \xrightarrow{g} Y\ ) = F(g)$. Furthermore, by proposition 5.27, $X \xleftarrow{f} W_1 = W_1$ and $X = X \xrightarrow{f} W_1$ are inverses of each other and by hypothesis $F(f)$ is invertible. So,

$$F'(\ X \xleftarrow{f} W_1 = W_1\ ) \circ F'(\ W_1 = W_1 \xrightarrow{f} X\ ) = F'(id_{\mathcal{D}/\mathcal{C}}) = F(id_{\mathcal{D}}) = id_{\mathcal{T}}.$$

Since, by the previous observation, $F'(\ W_1 = W_1 \xrightarrow{f} X\ ) = F(f)$, unicity of the inverse implies $F'(\ X \xleftarrow{f} W_1 = W_1\ ) = F(f)^{-1}$. Thus, $F' = \tilde{F}$.

□

Remark 5.34. It follows from proposition 5.33 that F_{univ} is universal in the sense defined in theorem 5.11. Indeed, let $\mathcal{T}$ be some triangulated category and let $F : \mathcal{D} \longrightarrow \mathcal{T}$ be a triangulated functor such that $\mathcal{C} \subseteq \mathrm{Ker}(F)$. We claim that F must take morphisms in $Mor_{\mathcal{C}}$ to isomorphisms. Let $f : X \longrightarrow Y$ be in $Mor_{\mathcal{C}}$ and consider $X \xrightarrow{f} Y \longrightarrow Z \longrightarrow T(X)$ a distinguished triangle in $\mathcal{D}$ extending f. Since $f \in Mor_{\mathcal{C}}$, $Z \in C$. F is a triangulated functor, so $F(X) \xrightarrow{F(f)} F(Y) \longrightarrow F(Z) \longrightarrow T(F(X))$ is distinguished in $\mathcal{T}$. Since $\mathcal{C} \subseteq \mathrm{Ker}(F)$, $F(Z) \cong 0$ and by proposition 3.25 $F(f)$ is an isomorphism. Applying proposition 5.33, we deduce that F factors uniquely through F_{univ}.

Let us briefly summarise what we have done and what is still left to do on the way to the proof of theorem 5.11. We have defined the Verdier quotient $\mathcal{D}/\mathcal{C}$ as the category whose objects are the same as those of $\mathcal{D}$ and whose morphisms are given as composition of morphisms of $\mathcal{D}$ with formal inverses of morphisms in $Mor_{\mathcal{C}}$. We have checked that this defined a category and understood that all morphisms in $Mor_{\mathcal{C}}$ are invertible in the quotient. Then, we defined the functor F_{univ} in a very natural way that resembles an inclusion. We have given a criterion for two morphisms in $\mathcal{D}$ to have the same F_{univ}-image and explained how we can lift statements to the quotient via F_{univ}. Finally, we verified that this functor is indeed universal. We are now at the point where all elements appearing in theorem 5.11 are defined and the part on the universality of the functor is shown. It remains to prove that the quotient is also a triangulated category, in particular an additive category, and that F_{univ} is a triangulated functor. Let us first turn our attention to the additive structure of the quotient.

5.2.4 Additive structure of the Verdier quotient

Theorem 5.35. *The category ${}^{\mathcal{D}}\!/_{\mathcal{C}}$ is an additive category. Moreover, F_{univ} is an additive functor.*

Proof. We need to check the following properties:
Existence of a zero-object: The object $0 \in \mathcal{D}$ defines an initial and terminal object in the quotient as well. Indeed, $X \xlongequal{\quad} X \longrightarrow 0$ is a morphism from X to 0 for any object X in the quotient. We see that it is unique as any other morphism $X \longrightarrow 0$, say $X \xleftarrow{f} P \longrightarrow 0$ is equivalent to the previous via:

$$\begin{tikzcd}
 & X \arrow[dl, equal] \arrow[d, equal] \arrow[dr] & \\
X \arrow[r, equal] & X \arrow[r] \arrow[d, "f"] & 0 \\
 & P \arrow[ul, "f"] \arrow[ur] &
\end{tikzcd}$$

By a symmetric argument we prove that 0 is also initial in the quotient.
Existence of finite products and coproducts: The direct sum in the quotient is the same as that in $\mathcal{D}$. It is left to the reader to check that this indeed satisfies the universal properties. Else, a proof can be found in [16], lemma 2.1.29.
Abelian group structure on the morphisms: We define the addition of two morphisms $f, g : X \longrightarrow Y$ as the composition $X \longrightarrow X \oplus X \xrightarrow{f \oplus g} Y \oplus Y \longrightarrow Y$. We need to check that there is an additive inverse. We use the fact that any morphism in the quotient can be written as $F_{univ}(g) \circ F_{univ}(f)^{-1}$ for $g \in \mathcal{D}$ and $f \in Mor_{\mathcal{C}}$ and that addition in $\mathcal{D}$ is well-defined. Denote by $-g$ the additive inverse of g in $\mathcal{D}$. Then, the additive inverse in the quotient is given by $F_{univ}(-g) \circ F_{univ}(f)^{-1}$ because

$$\begin{aligned} F_{univ}(-g) \circ F_{univ}(f)^{-1} + F_{univ}(g) \circ F_{univ}(f)^{-1} &= (F_{univ}(-g) + F_{univ}(g)) \circ F_{univ}(f)^{-1} = \\ F_{univ}(-g+g) \circ F_{univ}(f)^{-1} &= F_{univ}(0) \circ F_{univ}(f)^{-1} = 0. \end{aligned}$$

□

All that is left to do, is to show that the Verdier quotient is a triangulated category. As for the homotopy category, we will define the distinguished triangles in the Verdier quotient to be all those triangles isomorphic in the quotient to some nice, standard triangles. However, we do not yet have a good understanding of what it means for two objects to be isomorphic in the quotient. We need to fill this gap now, if we do not wish to run into difficulties later on.

5.2.5 Isomorphisms in Verdier quotient

The next proposition characterises what it means for a morphism of the quotient to be equivalent to the identity.

Proposition 5.36. *If in the quotient a morphism $X \xleftarrow{\alpha} P \xrightarrow{f} X$ is in the equivalence class of the identity $X \xlongequal{\quad} X \xlongequal{\quad} X$, then f is in $Mor_{\mathcal{C}}$.*

Proof. The morphism is in the same equivalence class as the identity if and only if the following diagram commutes for some object $W \in \mathcal{D}$ and morphisms $u, u' : W \longrightarrow X$

$$\begin{tikzcd}
 & X \arrow[dl, equal] \arrow[dr, equal] & \\
X & W \arrow[l, "u"] \arrow[r, "u"] \arrow[u, "u"] \arrow[d, "u'"] & X \\
 & P \arrow[ul, "\alpha"] \arrow[ur, "f"'] &
\end{tikzcd}$$

By proposition 5.23, $u, u' \in Mor_{\mathcal{C}}$. By commutativity of the diagram $f \circ u' = u$, so $f \in Mor_{\mathcal{C}}$. $\square$

Using this proposition, we can give an equivalent characterisation of isomorphisms in the quotient.

Proposition 5.37. *A morphism* $X \xleftarrow{\alpha} Z \xrightarrow{g} Y$ *in the quotient is invertible if and only if there exist morphisms* $f : X' \to X$ *and* $h : Y \to Y'$ *in* $\mathcal{D}$ *such that* $g \circ f$ *and* $h \circ g$ *are both in* $Mor_{\mathcal{C}}$.

Proof. Suppose such morphisms f and h exist. Since $g \circ f$ and $h \circ g$ are both in $Mor_{\mathcal{C}}$, $F_{univ}(g \circ f)$ and $F_{univ}(h \circ g)$ are invertible. This implies that $F_{univ}(g)$ has a right and left inverse. Indeed, by functoriality, $F_{univ}(h \circ g)^{-1} \circ F_{univ}(h) \circ F_{univ}(g) = id_Z$, so $F_{univ}(h \circ g)^{-1} \circ F_{univ}(h)$ is a left inverse of $F_{univ}(g)$. The right inverse is obtained by similar reasoning on $F_{univ}(g \circ f)$. Observe that $[(Z, \alpha, g)] = [F_{univ}(g) \circ F_{univ}(\alpha)^{-1}]$. The latter is invertible if and only if $F_{univ}(g)$ is. So the sufficiency is proven.
Conversely, suppose $F_{univ}(g) \circ F_{univ}(\alpha)^{-1}$ is invertible. Then, $F_{univ}(g)$ must be invertible too. Let the morphism given by $Y \xleftarrow{\beta} Q \xrightarrow{f} Z$ be a right inverse to $F_{univ}(g)$ that we denote ϕ. By definition, the composition $\phi \circ F_{univ}(g)$ is given by $Y \xleftarrow{\beta} Q \xrightarrow{f} Z \xrightarrow{g} Y$ and, since ϕ is the right inverse of $F_{univ}(g)$, the composition is in the equivalence class of the identity. We conclude by proposition 5.36 that $g \circ f \in Mor_{\mathcal{C}}$. The existence of h is proven dually. $\square$

The two next propositions explain under which conditions an object is isomorphic to 0 in the quotient and give us a better understanding of $\mathrm{Ker}(F_{univ})$. In particular, we will see that those propositions imply that $\mathcal{C} \subseteq \mathrm{Ker}(F_{univ})$ as claimed by theorem 5.11.

Proposition 5.38. *For any* X *in* $\mathcal{D}$ *the unique map* $g : X \longrightarrow 0$ *becomes an isomorphism in the quotient if and only if there exists an object* Y *in* $\mathcal{D}$ *such that* $X \oplus Y$ *is in* $\mathcal{C}$. *In other words,* $F_{univ}(X) \cong 0$ *if and only if* X *is a direct summand of an object in* $\mathcal{C}$.

Proof. First, suppose $F_{univ}(X) \cong 0$. By proposition 5.37, we can find a morphism $h : 0 \longrightarrow T(Y)$ such that $h \circ g \in Mor_{\mathcal{C}}$. Let $X \xrightarrow{0} T(Y) \longrightarrow T(Z) \longrightarrow T(X)$ be a distinguished triangle in $\mathcal{D}$ extending $h \circ g$. By proposition 3.29, the triangle is split. More precisely, $T(Z) \cong T(Y) \oplus T(X)$. Since T is an additive automorphism, $Z \cong X \oplus Y$. Because $h \circ g \in Mor_{\mathcal{C}}$, by definition of $Mor_{\mathcal{C}}$, $T(Z)$ and then also Z must be in $\mathcal{C}$.
Conversely, suppose there exists Y such that $X \oplus Y \in \mathcal{C}$. We aim to prove that $g : X \longrightarrow 0$ is an isomorphism in the quotient by applying proposition 5.37. Set $f : 0 \longrightarrow X$ and $h : 0 \longrightarrow T(Y)$. As 0 is an initial object, the composition $g \circ f : 0 \longrightarrow 0$ is the identity. In particular, it is in $Mor_{\mathcal{C}}$. To see that $h \circ g$ is also in $Mor_{\mathcal{C}}$, observe that the same split triangle as before extends $h \circ g : X \longrightarrow T(Y)$. This time, $T(Y) \oplus T(X) \in \mathcal{C}$ by hypothesis, so $h \circ g \in Mor_{\mathcal{C}}$ as well. We conclude by proposition 5.37 that $F_{univ}(g)$ is an isomorphism. $\square$

Remark 5.39. It follows from proposition 5.38 that $\mathcal{C} \subseteq \mathrm{Ker}(F_{univ})$ as wanted by theorem 5.11. Remember that the objects of $\mathrm{Ker}(F_{univ})$ are the objects of $\mathcal{D}$ whose image is isomorphic to 0 in the quotient. Now we have a precise description of the kernel. It consists of exactly those objects that are direct summands of objects in $\mathcal{C}$. In particular, if $X \in \mathcal{C}$, then $X \oplus 0 \cong X \in \mathcal{C}$. By proposition 5.38, we have that $F_{univ}(X) \cong 0$, so X belongs to the kernel. Moreover, if $\mathcal{C}$ is thick, then $\mathcal{C} = \mathrm{Ker}(F_{univ})$. The attentive reader might have already suspected this structure, remembering that by 5.17 the kernel is a thick subcategory.

Proposition 5.40. *Consider* $g : X \longrightarrow Y$ *a morphism in* $\mathcal{D}$. $F_{univ}(g)$ *is an isomorphism in the quotient if and only if for every distinguished* $\mathcal{D}$*-triangle* $X \xrightarrow{g} Y \longrightarrow Z \longrightarrow T(X)$ *, the object* Z *is a direct summand of an object in* $\mathcal{C}$ *i.e. there exists* Z' *such that* $Z \oplus Z' \in \mathcal{C}$.

The proof is long and quite technical. Moreover, it doesn't bring any interesting ideas. It is therefore omitted. One can find the proof in [16], proposition 2.1.35.

Remark 5.41. This proposition deepens our understanding of $\mathrm{Ker}(F_{univ})$ some more. It shows that the kernel is the smallest thick subcategory containing $\mathcal{C}$.

We end this subsection with a proposition that gives a condition for two distinguished triangles in $\mathcal{D}$ to be isomorphic in the quotient. This will be very useful to prove that axiom TR5 holds in $\mathcal{D}/\mathcal{C}$ as it will enable us to translate statements from $\mathcal{D}$ to the quotient. It is interesting to note the similarity between the following proposition and proposition 3.22.

Proposition 5.42. *Consider the following commutative diagram where the rows are distinguished triangles in $\mathcal{D}$*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r] \arrow[d, "g"'] & Z \arrow[r] & T(X) \arrow[d, equal] \\
X \arrow[r, "g\circ f"'] & Y' \arrow[r] & Z' \arrow[r] & T(X)
\end{tikzcd}$$

Suppose $F_{univ}(g)$ is an isomorphism. Then, there exists a morphism $h : Z \longrightarrow Z'$ such that the diagram commutes and $F_{univ}(h)$ is an isomorphism as well.

Proof. By TR5, we get the following commutative diagram where the third row is a distinguished triangle extending g given by TR2:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r] \arrow[d, "g"] & Z \arrow[r] \arrow[d, dashed, "h"] & T(X) \arrow[d, equal] \\
X \arrow[r, "g\circ f"] \arrow[d, "f"'] & Y' \arrow[r] \arrow[d, equal] & Z' \arrow[r] \arrow[d, dashed] & T(X) \arrow[d, "T(f)"] \\
Y \arrow[r, "g"] \arrow[d] & Y' \arrow[r] \arrow[d] & Z'' \arrow[r] \arrow[d, equal] & T(Y) \arrow[d] \\
Z \arrow[r, dashed] & Z' \arrow[r, dashed] & Z'' \arrow[r] & T(Z)
\end{tikzcd}$$

Since $F_{univ}(g)$ is an isomorphism, applying proposition 5.40 we find that Z'' is a direct summand of an object in $\mathcal{C}$. Applying the same proposition, but the converse, to the last row, we find that $F_{univ}(h)$ is an isomorphism. $\square$

Remark 5.43. By TR3 in $\mathcal{D}$, we get a similar statement for

$$\begin{tikzcd}
X \arrow[r, "f\circ g"] \arrow[d, "g"] & Y \arrow[r] \arrow[d, equal] & Z \arrow[r] & T(X) \arrow[d, "T(g)"] \\
X' \arrow[r, "f"'] & Y \arrow[r] & Z' \arrow[r] & T(X')
\end{tikzcd}$$

5.2.6 Triangulated structure of the Verdier localisation

We finally have all the tools to see that the Verdier quotient is a triangulated category. We define the translation functor T on $\mathcal{D}/\mathcal{C}$ in the natural way: on the objects it is defined like on $\mathcal{D}$ and it sends the diagram $X \xleftarrow{f} W_1 \xrightarrow{g} Y$ to $T(X) \xleftarrow{T(f)} T(W_1) \xrightarrow{T(g)} T(Y)$. Observe that T and F_{univ} commute, so we can choose the natural transformation $\phi_{univ} : T \circ F_{univ} \longrightarrow F_{univ} \circ T$ required by definition 5.15 to be the identity.
We define the distinguished triangles in $\mathcal{D}/\mathcal{C}$ to be all those isomorphic in the quotient to triangles of the form $F_{univ}(X) \to F_{univ}(Y) \to F_{univ}(Z) \to T(F_{univ}(X))$ where $X \to Y \to Z \to T(X)$ is a distinguished triangle in $\mathcal{D}$.

Theorem 5.44. *With this triangulation, $\mathcal{D}/\mathcal{C}$ is a triangulated category. Moreover, it makes F_{univ} a triangulated functor.*

Proof. It is clear by definition that F_{univ} takes triangles to triangles, thus it is a triangulated functor.
Let us now turn to the triangulated structure of the quotient.
TR0 and TR1 are satisfied by definition. TR3 is also a direct consequence of TR3 for $\mathcal{D}$. The proof of TR4 can be omitted by proposition 3.35. We will first prove TR2 and then TR5.

TR2: Recall from proposition 5.30 that every morphism in the quotient can be written as the composition $F_{univ}(u) \circ F_{univ}(f)^{-1} : X \longrightarrow Y$ where $u : P \longrightarrow Y$ is a morphism in $\mathcal{D}$ and $f : P \longrightarrow X$ is a morphism in $Mor_{\mathcal{C}}$. Let $P \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(P)$ be a distinguished triangle in $\mathcal{D}$

extending u. We get the following isomorphism in the quotient:

$$\begin{tikzcd}
F_{univ}(X) \arrow[r, "F_{univ}(u)\circ F_{univ}(f)^{-1}"] \arrow[d, "F_{univ}(f)^{-1}"] & F_{univ}(Y) \arrow[r, "F_{univ}(v)"] \arrow[d, equal] & F_{univ}(Z) \arrow[r, "F_{univ}(w)"] \arrow[d, equal] & T(F_{univ}(X)) \arrow[d, "T(F_{univ}(u)\circ F_{univ}(f)^{-1})"] \\
F_{univ}(P) \arrow[r, "F_{univ}(u)"] & F_{univ}(Y) \arrow[r, "F_{univ}(v)"] & F_{univ}(Z) \arrow[r, "F_{univ}(w)"] & T(F_{univ}(Y))
\end{tikzcd}$$

The bottom line is distinguished since it is the F_{univ}-image of a distinguished triangle in $\mathcal{D}$. By the isomorphism, the upper row is a distinguished triangle extending $F_{univ}(u) \circ F_{univ}(f)^{-1}$.

TR5: It solely remains to prove TR5. In that aim let

$$A \xrightarrow{f} B \longrightarrow C \longrightarrow T(A)$$

$$A \xrightarrow{g\circ f} \tilde{B} \longrightarrow \tilde{C} \longrightarrow T(A)$$

$$B \xrightarrow{g} \tilde{B} \longrightarrow C' \longrightarrow T(B)$$

be three distinguished triangles in ${}^{\mathcal{D}}\!/_{\mathcal{C}}$.

We want to get the following commutative diagram such that the bottom row is a distinguished triangle:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, dashed, "\xi_1"] & T(A) \arrow[d, equal] \\
A \arrow[r, "g\circ f"] \arrow[d, "f"] & \tilde{B} \arrow[r] \arrow[d, equal] & \tilde{C} \arrow[r] \arrow[d, dashed, "\xi_2"] & T(A) \arrow[d, "T(f)"] \\
B \arrow[r, "g"] \arrow[d] & \tilde{B} \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d, equal] & T(B) \arrow[d] \\
C \arrow[r, dashed, "\xi_1"] & \tilde{C} \arrow[r, dashed, "\xi_2"] & C' \arrow[r] & T(C)
\end{tikzcd} \tag{22}$$

All arrows except the dashed ones are already given. We will find ξ_1 and ξ_2, lifting the diagram to $\mathcal{D}$ and using TR5 there.

By proposition 5.32, we find a square $\begin{tikzcd} \alpha \arrow[r, "f'"] \arrow[d, equal] & \beta \arrow[d, "g'"] \\ \alpha \arrow[r, "g'\circ f'"] & \tilde{\beta} \end{tikzcd}$ which commutes in $\mathcal{D}$ and whose F_{univ}-image is isomorphic (in the quotient) to $\begin{tikzcd} A \arrow[r, "f"] \arrow[d, equal] & B \arrow[d, "g"] \\ A \arrow[r, "g\circ f"] & \tilde{B} \end{tikzcd}$

By TR2 for $\mathcal{D}$, we can extend the two lines of the square in $\mathcal{D}$ to distinguished triangles. We get:

$$\begin{tikzcd}
\alpha \arrow[r] \arrow[d, equal] & \beta \arrow[r] \arrow[d] & \gamma \arrow[r] & T(\alpha) \arrow[d, equal] \\
\alpha \arrow[r] & \tilde{\beta} \arrow[r] & \tilde{\gamma} \arrow[r] & T(\alpha)
\end{tikzcd}$$

where both lines are distinguished triangles.

We can also extend g' to a $\mathcal{D}$-distinguished triangle $\beta \xrightarrow{g'} \tilde{\beta} \longrightarrow \gamma' \longrightarrow T(\beta)$

By TR5 for $\mathcal{D}$, we get the dashed arrows in

$$\begin{tikzcd}
\alpha \arrow[r, "f'"] \arrow[d, equal] & \beta \arrow[r] \arrow[d, "g'"'] & \gamma \arrow[r] \arrow[d, dashed, "\psi_1"] & T(\alpha) \arrow[d, equal] \\
\alpha \arrow[r, "g'\circ f'"] \arrow[d, "f'"'] & \tilde{\beta} \arrow[r] \arrow[d, equal] & \tilde{\gamma} \arrow[r] \arrow[d, dashed, "\psi_2"] & T(\alpha) \arrow[d, "T(f')"] \\
\beta \arrow[r, "g'"] \arrow[d] & \tilde{\beta} \arrow[r] \arrow[d] & \gamma' \arrow[r] \arrow[d, equal] & T(\beta) \arrow[d] \\
\gamma \arrow[r, dashed, "\psi_1"] & \tilde{\gamma} \arrow[r, dashed, "\psi_2"] & \gamma' \arrow[r] & T(\gamma)
\end{tikzcd}$$

where the bottom row is also distinguished.

By functoriality, the diagram still commutes if we apply F_{univ} and by definition of distinguished triangles in the quotient the bottom row will remain distinguished.

Moreover, the F_{univ}-image of the top left square being isomorphic to the top left square in (22) by proposition 5.42 we can extend

$$\begin{tikzcd}
F_{univ}(\alpha) \arrow[r] \arrow[d] & F_{univ}(\beta) \arrow[r] \arrow[d] & F_{univ}(\gamma) \arrow[r] & T(F_{univ}(\alpha)) \arrow[d] \\
A \arrow[r] & B \arrow[r] & C \arrow[r] & T(A)
\end{tikzcd}$$

where the vertical arrows are $\mathcal{D}/\mathcal{C}$-isomorphisms to an isomorphism of triangles in the quotient

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, dashed, "\phi_1"] & T(A) \arrow[d] \\
F_{univ}(\alpha) \arrow[r] & F_{univ}(\beta) \arrow[r] & F_{univ}(\gamma) \arrow[r] & T(F_{univ}(\alpha))
\end{tikzcd}$$

Similarly, we get an isomorphism of triangles

$$\begin{tikzcd}
F_{univ}(\alpha) \arrow[r] \arrow[d] & F_{univ}(\tilde{\beta}) \arrow[r] \arrow[d] & F_{univ}(\tilde{\gamma}) \arrow[r] \arrow[d, dashed, "\phi_2"] & T(F_{univ}(\alpha)) \arrow[d] \\
A \arrow[r] & \tilde{B} \arrow[r] & \tilde{C} \arrow[r] & T(A)
\end{tikzcd}$$

Combining these we get a map $C \longrightarrow \tilde{C}$ given by $\phi_2 \circ F_{univ}(\psi_1) \circ \phi_1$.

Similarly, notice the F_{univ}-image of $\begin{tikzcd} \alpha \arrow[r, "g'\circ f'"] \arrow[d, "f'"] & \tilde{\beta} \arrow[d, equal] \\ \beta \arrow[r, "g'"] & \tilde{\beta} \end{tikzcd}$ is isomorphic to $\begin{tikzcd} A \arrow[r, "g\circ f"] \arrow[d, "f"] & \tilde{B} \arrow[d, equal] \\ B \arrow[r, "g"] & \tilde{B} \end{tikzcd}$.

As above, we can construct an isomorphism of triangles

$$\begin{tikzcd}
F_{univ}(\beta) \arrow[r] \arrow[d] & F_{univ}(\tilde{\beta}) \arrow[r] \arrow[d] & F_{univ}(\gamma') \arrow[r] \arrow[d, dashed, "\phi_3"] & T(F_{univ}(\beta)) \arrow[d] \\
B \arrow[r] & \tilde{B} \arrow[r] & \tilde{C} \arrow[r] & T(A)
\end{tikzcd}$$

We get a map $\tilde{C} \longrightarrow C'$ given by $\phi_3 \circ F_{univ}(\psi_2) \circ \phi_2^{-1}$.

It is clear that the choices $\xi_1 = \phi_2 \circ F_{univ}(\psi_1) \circ \phi_1$ and $\xi_2 = \phi_3 \circ F_{univ}(\psi_2) \circ \phi_2^{-1}$ let diagram (22) commute.

Moreover,

$$\begin{tikzcd}
C \arrow[r, "\xi_1"] \arrow[d, "\phi_1"] & \tilde{C} \arrow[r, "\xi_2"] \arrow[d, "\phi_2^{-1}"] & C' \arrow[r] \arrow[d, "\phi_3"] & T(C) \arrow[d, "T(\phi_1)"] \\
F_{univ}(\gamma) \arrow[r, "F_{univ}(\psi_1)"] & F_{univ}(\tilde{\gamma}) \arrow[r, "F_{univ}(\psi_2)"] & F_{univ}(\gamma') \arrow[r] & T(F_{univ}(\gamma))
\end{tikzcd}$$

exhibits an isomorphism, confirming that the upper row of the above diagram (the lowest row in TR5) is really a distinguished triangle. Hence, $\mathcal{D}/\mathcal{C}$ is a triangulated category.

$\square$

We have now proven theorem 5.11. Indeed, for any triangulated category $\mathcal{D}$ and any triangulated subcategory $\mathcal{C}$, we constructed the Verdier quotient $\mathcal{D}/\mathcal{C}$. It is the category whose objects are the objects of $\mathcal{D}$ and whose morphisms are equivalence classes of diagrams $X \xleftarrow{f} Z \xrightarrow{g} Y$ with $f \in Mor_{\mathcal{C}}$ (c.f def. 5.22). We have checked that this defines a category and have defined an additive (thm. 5.35) and triangulated structure (thm. 5.44) on it. We have defined F_{univ} (def. 5.29) and verified it is a triangulated functor between $\mathcal{D}$ and $\mathcal{D}/\mathcal{C}$. In remark 5.39, we have seen that $\mathcal{C} \subseteq \mathrm{Ker}(F_{univ})$ and in remark 5.34 we explained why any triangulated functor that contains $\mathcal{C}$ in its kernel factors uniquely through F_{univ}.
Observe that the Verdier localisation allows us to construct many more triangulated categories. As soon as we have two triangulated categories and a triangulated functor between them, we can define the Verdier localisation at the kernel. If we have a triangulated category with a triangulated subcategory, we can construct a new category, their Verdier quotient. By theorem 5.11, this gives rise to a new triangulated category.
This concludes the excursion to the general setting on Verdier localisation and brings us back to the derived category of some abelian category.

5.3 Back to the derived category

Recall that for an abelian category with translation $(\mathcal{A}, T)$ we would like to construct a new category from $K_c(\mathcal{A})$ via Verdier localisation, in which all quasi-isomorphisms are invertible. One question remains: which subcategory of $K_c(\mathcal{A})$ do we choose to construct the derived category as Verdier quotient?

Definition 5.45. We denote by $\mathcal{N}$ the full subcategory of $K_c(\mathcal{A})$ whose objects are given by those objects $X \in K_c(\mathcal{A})$ such that $H(X) \cong 0$ where H denotes the cohomology functor.

We want to localise $K_c(\mathcal{A})$ at $\mathcal{N}$. Before we can apply Verdier localisation we need to check that $\mathcal{N}$ is a triangulated subcategory. Indeed, even though $\mathcal{N}$ is the kernel of the cohomology functor, we cannot conclude yet because H is not a triangulated functor.

Proposition 5.46. *$\mathcal{N}$ is a triangulated subcategory of $K_c(\mathcal{A})$. The corresponding system $Mor_{\mathcal{N}}$ consists of all quasi-isomorphisms.*

Proof. It is easy to check that $\mathcal{N}$ is full, additive and closed under T. We need to verify the conditions on the triangles. Suppose $X \xrightarrow{f} Y \longrightarrow Z \longrightarrow T(X)$ is a distinguished triangle in $K_c(\mathcal{A})$ with $X, Y \in \mathcal{N}$. We want Z to be in $\mathcal{N}$ as well. Recall from corollary 5.9 that H is cohomological, so it gives rise to a long exact sequence $\cdots \longrightarrow H(X) \xrightarrow{H(f)} H(Y) \longrightarrow H(Z) \longrightarrow H(T(X)) \longrightarrow \cdots$
By exactness and since $X, Y \in \mathcal{N}$, so $H(Y) \cong H(T(X)) \cong 0$, it follows that $H(Z) \cong 0$ as wanted.
For the second part of the statement, suppose $f : X \longrightarrow Y$ is in $Mor_{\mathcal{N}}$. This means that in the above triangle the object $Z \in \mathcal{N}$. So in the long exact sequence arising through H

$$\cdots \longrightarrow H(T^{-1}(Z)) \longrightarrow H(X) \xrightarrow{H(f)} H(Y) \longrightarrow H(Z) \longrightarrow \cdots$$

$H(T^{-1}(Z))$ and $H(Z)$ are isomorphic to zero. By proposition 2.38, it follows that $H(f)$ is an isomorphism. A similar argument shows that any quasi-isomorphism must be in $Mor_{\mathcal{N}}$. Thus, $Mor_{\mathcal{N}}$ contains all quasi-isomorphisms and only them. $\square$

Remark 5.47. This theorem gives us a nice characterisation of quasi-isomorphisms in $K_c(Gr(Ab))$. It tells us that a morphism is a quasi-isomorphism if and only if its mapping cone is an exact chain complex. Indeed, by the theorem, quasi-isomorphisms are exactly the morphisms in $Mor_{\mathcal{N}}$. This means that if $f: X \longrightarrow Y$ is a quasi-isomorphism, the triangle extending f $\;X \xrightarrow{f} Y \longrightarrow Mc(f) \longrightarrow T(X)$ has the property that $Mc(f)$ is in $\mathcal{N}$, that is, $H(Mc(f)) = 0$. In $K_c(Gr(Ab))$ this is equivalent to the mapping cone being an exact sequence. This is a result that can be easily proved by hand, without having to use the general abstract framework of triangulated categories.

Remark 5.48. Actually, the previous theorem holds in a more general context for any cohomological functor $H: \mathcal{D} \longrightarrow \mathcal{A}$, where $\mathcal{D}$ is any triangulated category and $\mathcal{A}$ is some abelian category. The general statement is due to Verdier:

Theorem 5.49. *Given a cohomological functor $H: \mathcal{D} \longrightarrow \mathcal{A}$, the collection of morphisms in $\mathcal{D}$ mapped to isomorphisms by H is equal to $Mor_{\mathcal{C}}$ where $\mathcal{C}$ is the full triangulated subcategory of $\mathcal{D}$ consisting of objects $\{X \in \mathcal{D} : H(X) \cong 0\}$.*

The proof can be found in [17], proposition 2.1.17.

We are at last ready to define the derived category.

Definition 5.50. The derived category of some abelian category $\mathcal{A}$, denoted by $\mathcal{D}(\mathcal{A})$, is the Verdier localisation $K_c(\mathcal{A})/\mathcal{N}$.

By theorem 5.11, this is a triangulated category. The distinguished triangles are those isomorphic in the derived category to the F_{univ}-image of a mapping cone triangle in $K_c(\mathcal{A})$. During the construction of the Verdier localisation we have seen that any element in $Mor_{\mathcal{N}}$ is invertible in the quotient. Thus, in the derived category all quasi-isomorphisms become isomorphisms as wanted. Therefore, there are a lot more isomorphisms in the derived category than in $K_c(\mathcal{A})$, so, as a consequence of TR0, there are also many more distinguished triangles. Recall that at the end of section 4 we had discussed that in the homotopy category one could find exact sequences that cannot be extended to a distinguished triangle. Now that we have many more triangles, we can hope this correspondence no longer fails. And indeed, as claimed earlier, it won't.

Theorem 5.51. *Let $\;0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0\;$ be an exact sequence in $\mathcal{A}_c$. Then, there exists a triangle $\;X \xrightarrow{F_{univ}(f)} Y \xrightarrow{F_{univ}(g)} Z \longrightarrow T(X)\;$ that is distinguished in $\mathcal{D}(\mathcal{A})$. Moreover, Z is isomorphic in $\mathcal{D}(\mathcal{A})$ to $Mc(f)$, the mapping cone of f.*

Proof. Consider the following commutative diagram in $\mathcal{A}_c$ with exact rows

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, equal] \arrow[d, equal] & X \arrow[r, "0"] \arrow[d, "f"] & 0 \arrow[r] \arrow[d, "i"] & 0 \\
0 \arrow[r] & X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z \arrow[r] & 0
\end{tikzcd}$$

This induces an exact sequence in $\mathcal{A}_c$ on the mapping cones $\;0 \longrightarrow Mc(id_X) \xrightarrow{id \oplus f} Mc(f) \xrightarrow{0 \oplus g} Mc(i) \longrightarrow 0\;$.

By theorem 5.8, we get an exact sequence $\;H(Mc(id_X)) \longrightarrow H(Mc(f)) \longrightarrow H(Mc(i)) \longrightarrow H(T(Mc(id)))\;$.

By remark 5.47, $H(Mc(id_X)) = 0$ because id_X is an isomorphism so also a quasi-isomorphism. Thus, the exact sequence becomes $\;0 \longrightarrow H(Mc(f)) \longrightarrow H(Mc(i)) \longrightarrow 0\;$. Using proposition 2.38, this implies $H(0 \oplus g)$ is an isomorphism, so $0 \oplus g$ is a quasi-isomorphism. $Mc(i) = T(0) \oplus Z \cong Z$ so $0 \oplus g$ defines a quasi-isomorphism between the mapping cone of f and Z. Since quasi-isomorphisms are isomorphisms in the derived category, we get the second part of the statement.
To show that the triangle given in the theorem is really distinguished, consider the following diagram:

$$\begin{tikzcd}
X \arrow[r, "F_{univ}(f)"] \arrow[d, equal] & Y \arrow[r, "F_{univ}(g)"] \arrow[d, equal] & Z \arrow[r, "F_{univ}(\omega)"] & T(X) \arrow[d, equal] \\
X \arrow[r, "F_{univ}(f)"'] & Y \arrow[r, "F_{univ}(\alpha(f))"'] & Mc(f) \arrow[r, "F_{univ}(\beta(f))"'] \arrow[u, "F_{univ}(0 \oplus g)"] & T(X)
\end{tikzcd}$$

where we define $F_{univ}(\omega) = F_{univ}(\beta(f)) \circ F_{univ}(0 \oplus g)^{-1}$. The latter is well-defined as we showed that $0 \oplus g$ is a quasi-isomorphism. The bottom triangle is distinguished by definition of triangles in the derived category and the diagram exhibits an isomorphism. Thus, the upper triangle is distinguished too. $\square$

This correspondence between exact sequences and distinguished triangles should be added to the list of arguments why distinguished triangles should be interpreted as a generalisation of exact sequences to non-abelian settings discussed in section 3.3.

6 Limitations of the triangulated structure

We have now seen many interesting properties of triangulated categories and discussed two well-known examples, the homotopy category and the derived category. We noticed that in the latter there is a correspondence between distinguished triangles in the derived category and exact sequences in the initial one. However, we have not yet given examples of categories that are not triangulated or wondered if triangulated categories can be abelian. This will be done here.

6.1 Abelian versus triangulated categories

In this subsection, we explore the relation between abelian and triangulated categories, which will be surprisingly restrictive! Indeed, it turns out that an abelian category admits a triangulation if and only if it is semi-simple and in that case the triangulation is unique. The results presented here were sketched as an exercise in [5] and some remarks come from [8]. We begin by defining semisimple.

Definition 6.1. An abelian category is said to be semisimple if every exact triple splits. More precisely, if any short exact sequence $0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$ is isomorphic to $0 \longrightarrow X \xrightarrow{\iota_X} X \oplus Z \xrightarrow{\pi_Z} Z \longrightarrow 0$ where the maps are the inclusion and projection given by the universal property of $\oplus$.

Example 6.2. $\mathbf{Vect}_k$, the category of vector spaces over a field k, is semisimple. This will follow for example from theorem 6.3 and proposition 3.11, although there are much simpler proofs. However, this is not the case for the category of abelian groups since $0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$ is an exact sequences which cannot split (see remark 3.32).

Theorem 6.3. *Any category that is both triangulated and abelian is semisimple. Moreover, any distinguished triangle is isomorphic to a triangle of the form*

$$X \xrightarrow{f} Y \xrightarrow{\iota_{coker(f)}} T(ker(f)) \oplus coker(f) \xrightarrow{\pi_{T(ker(f))}} T(X)$$

where $\iota_{coker(f)} = \iota \circ k$ with $k : Y \to coker(f)$ the map that comes with the universal property of the cokernel and $\iota : coker(f) \to T(ker(f)) \oplus coker(f)$ is the inclusion given by the universal property of the direct sum and $\pi_{T(ker(f))} = l \circ p$ where $p : T(ker(f)) \oplus coker(f) \to T(ker(f))$ is the projection given by the universal property of the direct sum and $l : T(ker(f)) \to T(X)$ is the map coming with the universal property of the kernel.

Proof. Let $0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$ be a short exact sequence. We want to show that it splits. By corollary 2.37, f is a monomorphism. By theorem 3.31, f splits. In particular, there exists a retraction $r : Y \longrightarrow X$ such that $r \circ f = id_X$. By proposition 2.40, this is equivalent to the sequence being split.

For the second part, consider any distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X)$. Notice that we have exact sequences $0 \longrightarrow \ker(f) \longrightarrow X \xrightarrow{f} \operatorname{im}(f) \longrightarrow 0$ and $0 \longrightarrow \operatorname{im}(f) \longrightarrow Y \longrightarrow \operatorname{coker}(f) \longrightarrow 0$. Since, as was just proven, the category is semisimple, both sequences split. Hence, we get isomorphisms $\psi : X \to \ker(f) \oplus \operatorname{im}(f)$ and $\phi : Y \to \operatorname{im}(f) \oplus \operatorname{coker}(f)$. We have the following isomorphism of triangles:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\psi"'] & Y \arrow[r, "g"] \arrow[d, "\phi"'] & Z \arrow[r, "h"] \arrow[d, equal] & T(X) \arrow[d, equal] \\
\ker(f) \oplus \operatorname{im}(f) \oplus 0 \arrow[r, "{(0,id,i)}"'] & 0 \oplus \operatorname{im}(f) \oplus \operatorname{coker}(f) \arrow[r, "g \circ \phi^{-1}"'] & Z \arrow[r, "\tilde{h}"'] & T(X)
\end{tikzcd}$$

where i denotes the unique map $0 \longrightarrow \operatorname{coker}(f)$.

By TR1, $\operatorname{im}(f) = \operatorname{im}(f) \longrightarrow 0 \longrightarrow T(\operatorname{im}(f))$ is a distinguished triangle. We have already seen (prop. 3.12), that $0 \xrightarrow{i} \operatorname{coker}(f) = \operatorname{coker}(f) \longrightarrow 0$ is distinguished. Thus, by proposition 3.27, the direct sum of those triangles, $\operatorname{im}(f) \oplus 0 \xrightarrow{(id,i)} \operatorname{im}(f) \oplus \operatorname{coker}(f) \xrightarrow{(0,id)} 0 \oplus \operatorname{coker}(f) \longrightarrow T(\operatorname{im}(f)) \oplus 0$ is distinguished as well. Also, $\ker(f) \longrightarrow 0 \longrightarrow T(\ker(f)) = T(\ker(f))$ is distinguished. This can be seen by applying TR3 (the "only if" direction) twice to $T(\ker(f)) = T(\ker(f)) \longrightarrow 0 \longrightarrow T^2(\ker(f))$.

Thus, $\ker(f)\oplus \mathrm{im}(f)\oplus 0 \xrightarrow{(0,id,i)} 0\oplus \mathrm{im}(f)\oplus \mathrm{coker}(f) \longrightarrow T(\ker(f))\oplus 0\oplus \mathrm{coker}(f) \longrightarrow T(\ker(f))\oplus T(\mathrm{im}(f))\oplus 0$ is distinguished. We have the following commutative diagram:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\psi"] & Y \arrow[r, "g"] \arrow[d, "\phi"] & Z \arrow[r, "h"] \arrow[d, dashed] & T(X) \arrow[d, "T(\psi)"] \\
\ker(f)\oplus \mathrm{im}(f)\oplus 0 \arrow[r, "(0,id,i)"] & 0\oplus \mathrm{im}(f)\oplus \mathrm{coker}(f) \arrow[r] & T(\ker(f))\oplus 0\oplus \mathrm{coker}(f) \arrow[r] & T(\ker(f))\oplus T(\mathrm{im}(f))\oplus 0
\end{tikzcd}$$

The dashed arrow exists by TR4 and is an isomorphism by proposition 3.22. The result then follows from the observation that $T(\ker(f))\oplus 0\oplus \mathrm{coker}(f)\cong T(\ker(f))\oplus \mathrm{coker}(f)$ and, since by definition T is an additive functor, by proposition 2.23, $T(\ker(f))\oplus T(\mathrm{im}(f))\oplus 0\cong T(\ker(f)\oplus \mathrm{im}(f))$. ☐

We also get a converse to this theorem:

Theorem 6.4. *Any semisimple abelian category is triangulated. The triangulation is given by all triangles of the form* $X \xrightarrow{f} Y \xrightarrow{\iota_{coker(f)}} T(ker(f))\oplus coker(f) \xrightarrow{\pi_{T(ker(f))}} T(X)$ *and those isomorphic to them. The maps* $\iota_{coker(f)}$ *and* $\pi_{T(ker(f))}$ *are as in theorem 6.3.*

To prove this theorem, we will need the following lemma.

Lemma 6.5. *In a semisimple abelian category, the following are equivalent:*

i) The sequence

$$T^{-1}(Z) \xrightarrow{T^{-1}(h)} X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X) \xrightarrow{T(f)} T(Y)$$

is exact.

ii) The sequence

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X)$$

is isomorphic to a sequence of the form

$$X \xrightarrow{f} Y \xrightarrow{\iota_{coker(f)}} T(ker(f))\oplus coker f \xrightarrow{\pi_{T(ker(f))}} T(X) \tag{23}$$

The maps $\iota_{coker(f)}$ *and* $\pi_{T(Kerf)}$ *are as in theorem 6.3.*

Proof. An easy calculation shows that

$$T^{-1}(T(\ker(f))\oplus \mathrm{coker}(f)) \xrightarrow{T^{-1}(\pi)} X \xrightarrow{f} Y \xrightarrow{\iota} T(\ker(f))\oplus \mathrm{coker}(f) \xrightarrow{\pi} T(X) \xrightarrow{T(f)} T(Y)$$

is exact. Since a sequence isomorphic to an exact sequence is also exact (this was shown in the proof of TR0 in proposition 3.11), it follows that any sequence isomorphic to (23) satisfies i).
Conversely, let

$$T^{-1}(Z) \xrightarrow{T^{-1}(h)} X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X) \xrightarrow{T(f)} T(Y)$$

be an exact sequence. We can extract a short exact sequence $0 \longrightarrow \ker(h)=\mathrm{im}(g) \longrightarrow Z \longrightarrow \mathrm{coker}(g) \longrightarrow 0$. As we work over a semisimple category, this sequence is split, in particular there is an isomorphism $\psi : Z \to \mathrm{coker}(g)\oplus \ker(h)$. But by proposition 2.43, there are isomorphisms $\alpha : \ker(h) \to \mathrm{coker}(f)$ and $\beta : \mathrm{coker}(g) \to \ker(T(f))$. So, we get an isomorphism $(\beta\oplus\alpha)\circ\psi : Z \to \ker(T(f))\oplus \mathrm{coker}(f)$. This isomorphism induces an isomorphism on the exact sequences in the following way:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "g"] \arrow[d, equal] & Z \arrow[r, "h"] \arrow[d, "(\beta\oplus\alpha)\circ\psi"] & T(X) \arrow[d, equal] \\
X \arrow[r, "f"] & Y \arrow[r, "\iota_{\mathrm{coker}(f)}"] & T(\ker(f))\oplus \mathrm{coker}(f) \arrow[r, "\pi_{T(\ker(f))}"] & T(X)
\end{tikzcd} \tag{24}$$

To see that this diagram commutes, recall from the proof of proposition 2.38 that g factors uniquely through $\mathrm{im}(g)$. Moreover, $h\circ g=0$ by exactness so h factors uniquely through $\mathrm{coker}(g)$. So we can rewrite (24) as the following diagram:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r] \arrow[d, equal] \arrow[rr, bend left, "g"] & \mathrm{im}(g) \arrow[r] \arrow[d, equal] & Z \arrow[r] \arrow[d, "\psi"] \arrow[rr, bend left, "h"] & \mathrm{coker}(g) \arrow[r] \arrow[d, equal] & T(X) \arrow[d, equal] \\
X \arrow[r, "f"'] \arrow[d, equal] & Y \arrow[r] \arrow[d, equal] & \ker(h) \arrow[r, "\iota_{\ker(h)}"'] \arrow[d, "\alpha"] & \mathrm{coker}(g) \oplus \ker(h) \arrow[r, "\pi_{\mathrm{coker}(g)}"'] \arrow[d, "\beta\oplus\alpha"] & \mathrm{coker}(g) \arrow[r] \arrow[d, "\beta"] & T(x) \arrow[d, equal] \\
X \arrow[r, "f"'] & Y \arrow[r, "k"] \arrow[rr, bend right, "\iota_{\mathrm{coker}(f)}"'] & \mathrm{coker}(f) \arrow[r, "\iota"'] & T(\ker(f)) \oplus \mathrm{coker}(f) \arrow[r, "p"'] \arrow[rr, bend right, "\pi_{T(\ker(f))}"'] & T(\ker(f)) \arrow[r, "l"'] & T(X)
\end{tikzcd}$$

All lower squares commute by construction and definition of α and β (c.f. proof of 2.38). The upper rectangle commutes by definition of semisimplicity. Indeed, it assures that not only $Z \cong \mathrm{coker}(g)\oplus\ker(h)$, but also that ψ makes the upper diagram commute. Therefore, (24) really gives the isomorphism between the exact sequences that was claimed. $\square$

With this lemma in mind, we are in the best condition to prove that any semisimple abelian category is triangulated.

Proof. (theorem 6.4) We need to check that the triangulation defined in the theorem verifies all the axioms. TR0 and TR2 are clear by construction. TR1 is clear once one notices that $\ker(id)$=$\mathrm{coker}(id) = 0$ as was mentioned in remark 2.28. We won't prove TR4 as by proposition 3.35 it is implied by the other axioms.
We show the "if" statement of TR3, the "only if" statement being implied by proposition 3.36. Without loss of generality, it is sufficient to show that $Y \xrightarrow{-\pi_{\mathrm{coker}(f)}} T(\ker(f)) \oplus \mathrm{coker}(f) \xrightarrow{-\iota_{T(\ker(f))}} T(X) \xrightarrow{-T(f)} T(Y)$ is a distinguished triangle. Recall that, by the same argument as in the second part of theorem 6.3, $X \cong \ker(f) \oplus \mathrm{im}(f)$ and $T(X) \cong T(\ker(f)\oplus\mathrm{im}(f))$. Notice that by definition of the image $\ker(\pi_{\mathrm{coker} f})$=$\mathrm{im}(f)$ and that $\mathrm{coker}(\pi_{\mathrm{coker}(f)})$=$T(\ker(f))\oplus 0$. So $T(X) \cong \mathrm{coker}(\pi_{\mathrm{coker}(f)}))\oplus T(\ker(\pi_{\mathrm{coker}(f)}))$, so the triangle is isomorphic to $Y \xrightarrow{-\pi_{\mathrm{coker}(f)}} T(\ker(f)) \oplus \mathrm{coker}(f) \xrightarrow{\iota_{\mathrm{coker}(\pi)}} \mathrm{coker}(\pi_{\mathrm{coker}(f)})) \oplus T(\ker(\pi_{\mathrm{coker}(f)})) \xrightarrow{\pi_{T(\ker(\pi))}} T(Y)$. This triangle is distinguished by definition, so we conclude by TR0 that the initial triangle is distinguished too.

At last, we prove TR5. We want the existence of the dashed arrows in the following square:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "\pi_{\mathrm{coker}(f)}"] \arrow[d, "g"] & T(\ker(f)) \oplus \mathrm{coker}(f) \arrow[r, "T(\iota_{\ker(f)})"] \arrow[d, dashed, "\alpha"] & T(X) \arrow[d, equal] \\
X \arrow[r, "g\circ f"] \arrow[d, "f"] & Z \arrow[r, "\pi_{\mathrm{coker}(g\circ f)}"] \arrow[d, equal] & T(\ker(g\circ f)) \oplus \mathrm{coker}(g\circ f) \arrow[r, "T(\iota_{\ker(g\circ f)})"] \arrow[d, dashed, "\beta"] & T(X) \arrow[d, "T(f)"] \\
Y \arrow[r, "g"] \arrow[d, "\pi_{\mathrm{coker}(f)}"] & Z \arrow[r, "\pi_{\mathrm{coker}(g)}"] \arrow[d, "\pi_{\mathrm{coker}(g\circ f)}"] & T(\ker(g)) \oplus \mathrm{coker}(g) \arrow[r, "T(\iota_{\ker(g)})"] \arrow[d, equal] & T(Y) \arrow[d, "T(\pi_{\mathrm{coker}(f)})"] \\
T(\ker(f)) \oplus \mathrm{coker}(f) \arrow[r, dashed, "\alpha"] & T(\ker(g\circ f) \oplus \mathrm{coker}(g\circ f) \arrow[r, dashed, "\beta"] & T(\ker(g) \oplus \mathrm{coker}(g) \arrow[r, "\gamma"] & T(T(\ker(f)) \oplus \mathrm{coker}(f))
\end{tikzcd}$$

By the universal property of the kernel, we get the maps:

$$\begin{tikzcd}
\ker(g\circ f) \arrow[r, "\iota_{\ker(g\circ f)}"] & X \arrow[r, shift left, "g\circ f"] \arrow[r, shift right, "0"'] & Y \\
& \ker(f) \arrow[ul, dashed, "\exists! j_1"] \arrow[u, "\iota_{\ker(f)}"] \arrow[ur] &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\ker(g) \arrow[r, "\iota_{\ker(g)}"] & Y \arrow[r, shift left, "g"] \arrow[r, shift right, "0"'] & Z \\
& \ker(g\circ f) \arrow[ul, dashed, "\exists! j_2"] \arrow[u, "f\circ\iota_{\ker(g\circ f)}"] \arrow[ur] &
\end{tikzcd}$$

Dually, for the cokernel:

$$\begin{tikzcd}
\mathrm{coker}(f) \arrow[dr, dashed, "\exists! q_1"'] & Z \arrow[l, "\pi_{\mathrm{coker}(f)}"'] \arrow[d, "\pi_{\mathrm{coker}(g\circ f)}\circ f"] & Y \arrow[l, shift right, "f"'] \arrow[l, shift left, "0"] \arrow[dl] \\
& \mathrm{coker}(g\circ f) &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathrm{coker}(g\circ f) \arrow[dr, dashed, "\exists! q_2"'] & Z \arrow[l, "\pi_{\mathrm{coker}(g\circ f)}"'] \arrow[d, "\pi_{\mathrm{coker}(g)}"] & X \arrow[l, shift right, "g\circ f"'] \arrow[l, shift left, "0"] \arrow[dl] \\
& \mathrm{coker}(g) &
\end{tikzcd}$$

We set $\alpha = \begin{pmatrix} T(j_1) & 0 \\ 0 & q_1 \end{pmatrix}$ and $\beta = \begin{pmatrix} T(j_2) & 0 \\ 0 & q_2 \end{pmatrix}$. By construction, these maps make the diagram commute and it is an easy calculation to check that the bottom line is exact and moreover $\ker(\alpha) = \mathrm{im}(T^{-1}(\gamma))$, $\mathrm{im}(\gamma) = \ker(T(\alpha))$. By lemma 6.5, we conclude that the bottom row is a distinguished triangle. $\square$

Remark 6.6. It follows from this theorem that in a semisimple abelian category, any triangle is isomorphic to a direct sum of the three "trivial triangles": $X = X \longrightarrow 0 \longrightarrow T(X)$, $0 \longrightarrow X = X \longrightarrow 0$ and $X \longrightarrow 0 \longrightarrow T(X) = T(X)$.
Indeed, we have just shown that any distinguished triangle is isomorphic to a triangle of the form $X \xrightarrow{f} Y \longrightarrow T(\ker(f)) \oplus \mathrm{coker}(f) \longrightarrow T(X)$ and since by semisimplicity $X \cong \ker(f) \oplus \mathrm{im}(f)$ and $Y \cong \mathrm{im}(f) \oplus \mathrm{coker}(f)$, we deduce that any distinguished triangle is isomorphic to a triangle of the form $\ker(f) \oplus \mathrm{im}(f) \longrightarrow \mathrm{im}(f) \oplus \mathrm{coker}(f) \longrightarrow T(\ker(f)) \oplus \mathrm{coker}(f)) \longrightarrow T(X)$ Remembering that by proposition 2.23 T preserves direct sums, we see that the above triangle is a direct sum of:

$$\ker(f) \longrightarrow 0 \longrightarrow T(\ker(f)) = T(\ker(f))$$

$$\mathrm{im}(f) = \mathrm{im}(f) \longrightarrow 0 \longrightarrow T(\mathrm{im}(f))$$

$$0 \longrightarrow \mathrm{coker}(f) = \mathrm{coker}(f) \longrightarrow 0.$$

So in some sense, this category does not contain any interesting (non-trivial) triangles.

Recall that in proposition 3.11 we had shown $\mathbf{Vect}_k$ is a triangulated category. We had used the fact, which is a special case of lemma 6.5, that any triangle was isomorphic to $X \xrightarrow{f} Y \xrightarrow{\pi} \ker(f) \oplus \mathrm{im}(g) \xrightarrow{\iota} X$. Notice that here $\mathrm{im}(g) \cong \mathrm{coker}(f)$ as $\mathrm{coker}(f) = Y/\mathrm{im}(f) = Y/\ker(g) \cong \mathrm{im}(g)$. The first equality is true by definition, the second by exactness of the triangle and the third is the first isomorphism theorem. So the triangulation we chose naturally was precisely the triangulation given by theorem 6.4. In particular, since $\mathbf{Vect}_k$ is an abelian category, we deduce from theorem 6.3 that $\mathbf{Vect}_k$ is semisimple. In proposition 3.19, we had shown that there was no other choice for a triangulation on $\mathbf{Vect}_k$. This fact can be seen as a corollary of the second part of theorem 6.3.

Many examples of categories that cannot be triangulated arise from these theorems. Any abelian category that is not semisimple cannot be triangulated. Moreover, these theorems are nice tools to understand why we had to pass to the homotopy category to find a triangulation on $\mathcal{A}_d$. In general, we cannot find a triangulation on $\mathcal{A}_d$, since it would need to be semisimple whenever $\mathcal{A}$ is abelian. Conversely, they explain why $K_d(A)$ is often not abelian. Let us look at two concrete examples.

Corollary 6.7. *$Gr(Ab)_c$, that is, the category of chain complexes on abelian groups as defined in example 4.3 is not triangulated.*

Proof. As was mentioned in example 4.3, $Gr(Ab)_c$ is an abelian category because Ab is. So if it were triangulated, it would have to be semisimple. This is not the case. Consider the exact sequence of chain complexes concentrated at degree zero $0 \longrightarrow \mathbb{Z}_\bullet \xrightarrow{\times 2} \mathbb{Z}_\bullet \longrightarrow \mathbb{Z}/2\mathbb{Z}_\bullet \longrightarrow 0$. As shown in remark 3.32, this cannot split. $\square$

Corollary 6.8. *$K_c(Gr(Ab))$ is not abelian.*

Proof. Suppose for the sake of contradiction that $K_c(Gr(Ab))$ is abelian. Since it is triangulated, by theorem 6.3, it has to be semisimple. The same sequence as in corollary 6.7 has to split in $K_c(Gr(Ab))$, that is $\mathbb{Z}$ is isomorphic up to homotopy to $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. However, since the complexes are concentrated at degree 0, all homotopies are trivial, so we get an actual isomorphism, but this impossible for the same reasons as above. $\square$

6.2 Arrow category of a triangulated category

We will now look at another surprising example of a category that is not triangulated. We will need the following definition.

Definition 6.9 (Arrow Category)**.** Consider a category $\mathcal{C}$. We define the arrow category of $\mathcal{C}$ denoted by $Arr(\mathcal{C})$ as the category whose objects are the morphisms in $\mathcal{C}$ and whose morphisms are defined as follows. Let $f : A \to B$, $g : A' \to B'$ be two morphisms in $\mathcal{C}$, then

$$Hom_{Arr(\mathcal{C})}(f, g) := \{(u, v),\ u : A \to A',\ v : B \to B' \mid g \circ u = v \circ f\}.$$

The identity of f is given by (id_A, id_B). Composition is defined component-wise. That is, for two morphisms $(u, v) : f \longrightarrow g$ and $(u', v') : g \longrightarrow h$, the composition will be $(u', v') \circ (u, v) = (u' \circ u, v' \circ v)$.

A morphisms $f \to g$ in $Arr(\mathcal{C})$ can be visualised as the following square:

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] & A' \arrow[d, "g"] \\ B \arrow[r, "v"'] & B' \end{tikzcd}$$

Then, given morphisms (u, v) and (u', v') composition is given by the outer rectangle of the following diagram: $\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] & A' \arrow[r, "u'"] \arrow[d, "g"'] & A'' \arrow[d, "h"] \\ B \arrow[r, "v"'] & B' \arrow[r] & B'' \end{tikzcd}$

Remark 6.10. In the literature, the arrow category is sometimes also referred to as the category of morphisms and denoted $Mor(\mathcal{C})$.

The arrow category is a very natural construction and it seems reasonable to hope that such a strong property as being triangulated would remain true in the arrow category. However, as the following theorem illustrates, this isn't the case. This result comes from a lecture taught by Paul Balmer on tensor triangulated gemometry [18].

Theorem 6.11. *Let $(\mathcal{D}, T)$ be a triangulated category. Then, the arrow category $Arr(\mathcal{D})$ of $\mathcal{D}$ is triangulated if and only if $\mathcal{D}$ contains only the zero-object.*

Proof. For the sake of contradiction, suppose $Arr(\mathcal{D})$ is triangulated. Let A be an arbitrary object in $\mathcal{D}$, denote $id_A : A \longrightarrow A$ its identity and $t : A \longrightarrow 0$ the unique morphism from A to 0.
Consider the morphism of $Arr(\mathcal{D})$ given by $\alpha = (id_A, t) : id_A \to t$. We claim that α is an epimorphism. Indeed, suppose there exist morphisms $\gamma_1 = (f_0, f_1)$, $\gamma_2 = (h_0, h_1) : t \longrightarrow g$ where $g : B \longrightarrow C$ is some morphism of $\mathcal{D}$ such that $\gamma_1 \circ \alpha = \gamma_2 \circ \alpha$. Thus, $f_0 \circ id_A = h_0 \circ id_A$ and $f_1 \circ t = h_1 \circ t$ since $\gamma_1 \circ \alpha = (f_0 \circ id_A, f_1 \circ t) = \gamma_2 \circ \alpha = (h_0 \circ id_A, h_1 \circ t)$. It follows that $f_0 = h_0$. Moreover, recall that f_1 and h_1 are morphisms in $\mathcal{D}$ from 0 to C. 0 being an initial object, there is a unique morphism from 0 to C. This implies that $f_1 = h_1$ and $\gamma_1 = \gamma_2$ as well. Hence, α is an epimorphism as claimed.
Recall from proposition 3.33 that in a triangulated category all epimorphisms split. Therefore, α admits a section $\beta : t \longrightarrow id_A$ satisfying $\alpha \circ \beta = id_t := (id_A, id_0)$. Write β as (k, s). As β is a section of α, we have $id_A \circ k = id_A$ and $t \circ s = id_0$. Moreover, by the commutativity condition on morphisms of the arrow category for β, $id_A \circ k = s \circ t$. Hence, id_A factors through 0 as $id_A = id_A \circ k = s \circ t$. This implies that $A = 0$. Since A had been chosen arbitrarily, all objects in $\mathcal{D}$ must be 0.

□

7 How many different triangulations can a category have?

We have just seen two examples of categories that do not admit any triagulations: the arrow category of a triangulated category and any abelian category that is not semisimple. We also encountered triangulated categories that admit a unique triangulation: any semisimple abelian category, in particular vector spaces. In definition 3.9, we introduced the notion of anti-distinguished triangles, which gave rise to a triangulation as well. We had claimed, without proof, that this triangulation can be different from the initial one. This showed that certain categories admit two different triangulations.
This naturally leads to the following question: how many distinct triangulations can a category admit? We will look at an example which shows that some categories admit infinitely many distinct triangulations and prove the claim on anti-distinguished triangles. This example was suggested in [2]. In order to do so, we need a few preliminary definitions and propositions.

Definition 7.1 (global endomorphism)**.** Let $(\mathcal{D}, T)$ be an additive category with translation. We call a natural transformation α a global endomorphism of $(\mathcal{D}, T)$ if it is an endomorphism of the identity functor, that is, a natural transformation from $id_{\mathcal{D}}$ to $id_{\mathcal{D}}$ that commutes with T. In other words, for any objects $A, B \in \mathcal{D}$, α defines morphisms $\alpha_A : A \longrightarrow A$ and $\alpha_B : B \longrightarrow B$ such that for any morphism of $\mathcal{D}$ $f : A \longrightarrow B$, $f \circ \alpha_A = \alpha_B \circ f$ and $\alpha_{T(A)} = T(\alpha_A)$.
We say α is a global automorphism if it is invertible i.e. α_A is invertible for any object $A \in \mathcal{D}$.

Definition 7.2. Let $(\mathcal{D}, T, \mathcal{T})$ be a triangulated category with triangulation $\mathcal{T}$. Let α be a global automorphism of $\mathcal{T}$. We define the class $\mathcal{T}_\alpha$ as the collection of all triangles $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$ such that the triangle $X \xrightarrow{u\circ\alpha_X} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$ is distinguished with respect to $\mathcal{T}$.

Remark 7.3. The triangle $X \xrightarrow{u\circ\alpha_X} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$ is distinguished with respect to $\mathcal{T}$ if and only if the triangle $X \xrightarrow{u} Y \xrightarrow{v\circ\alpha_Y} Z \xrightarrow{w} T(X)$ is distinguished with respect to $\mathcal{T}$ if and only if the triangle $X \xrightarrow{u} Y \xrightarrow{\alpha_Z\circ v} Z \xrightarrow{w} T(X)$ is distinguished with respect to $\mathcal{T}$ and so on i.e we can move α around freely.
Indeed, by definition of α, $u \circ \alpha_X = \alpha_Y \circ u$ and $v \circ \alpha_Y = \alpha_Z \circ v$, so we notice the above triangles are all isomorphic via the diagram:

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[rr, bend left, "u\circ\alpha_X"] \arrow[d, equal] & X \arrow[r, "u"] & Y \arrow[r, "v"] \arrow[d, equal] & Z \arrow[r, "w"] \arrow[d, equal] & T(X) \arrow[d, equal] \\
X \arrow[r, "u"] \arrow[rr, bend left, "\alpha_Y\circ u"] \arrow[d, equal] & Y \arrow[r, "\alpha_Y"] \arrow[d, equal] & Y \arrow[r, "v"] \arrow[d, equal] & Z \arrow[r, "w"] \arrow[d, equal] & T(X) \arrow[d, equal] \\
X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r, "\alpha_Y"] \arrow[rr, bend right, "v\circ\alpha_Y"'] \arrow[d, equal] & Y \arrow[r, "v"] & Z \arrow[r, "w"] \arrow[d, equal] & T(X) \arrow[d, equal] \\
X \arrow[r, "u"] & Y \arrow[r, "v"] \arrow[rr, bend right, "\alpha_Z\circ v"'] & Z \arrow[r, "\alpha_Z"] & Z \arrow[r, "w"] & T(X)
\end{tikzcd}$$

We conclude by TR0 that if one of the triangles is distinguished, all are.

Proposition 7.4. *For any global automorphism α, the category $(\mathcal{D}, T, \mathcal{T}_\alpha)$ is triangulated.*

Proof. We check each axiom separately. Each time, the main idea is to exploit the fact that $\mathcal{T}$ is a triangulation. As usual, by proposition 3.35, we don't have to verify TR4 as it is implied by the remaining axioms.

TR0: Notice that if φ, ψ, ϕ are isomorphisms in the following diagram

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "\varphi"'] & Y \arrow[r, "v"] \arrow[d, "\psi"'] & Z \arrow[r, "w"] \arrow[d, "\phi"'] & T(X) \arrow[d] \\
A \arrow[r, "a"] & B \arrow[r, "b"] & C \arrow[r, "c"] & T(C)
\end{tikzcd}$$

they are also isomorphisms for

$$\begin{tikzcd}
X \arrow[r, "u\circ\alpha_X"] \arrow[d, "\varphi"'] & Y \arrow[r, "v"] \arrow[d, "\psi"'] & Z \arrow[r, "w"] \arrow[d, "\phi"'] & T(X) \arrow[d] \\
A \arrow[r, "a\circ\alpha_A"] & B \arrow[r, "b"] & C \arrow[r, "c"] & T(C)
\end{tikzcd}$$

since $a \circ \varphi = \psi \circ u \implies a \circ \alpha_A \circ \varphi = \psi \circ u \circ \alpha_X$ as $\varphi \circ \alpha_X = \alpha_A \circ \varphi$. Suppose the upper row of the first diagram is in $\mathcal{T}_\alpha$. Then we want to show that so is the lower row. By definition of $\mathcal{T}_\alpha$, the upper row of the second diagram is a distinguished triangle with respect to $\mathcal{T}$. By TR0 for $\mathcal{T}$, we deduce that the lower row of the second diagram is distinguished as well. Thus, by definition of $\mathcal{T}_\alpha$, the lower row of the first diagram is distinguished with respect to $\mathcal{T}_\alpha$ as well.

TR1: By remark 7.3, $X = X \longrightarrow 0 \longrightarrow T(X) \in \mathcal{T}_\alpha \iff X = X \xrightarrow{0\circ\alpha_X} 0 \longrightarrow T(X) \in \mathcal{T}$. The latter is clear by TR1 for $\mathcal{T}$.

TR2: Observe that $u : X \longrightarrow Y$ can be completed to a distinguished triangle in $\mathcal{T}_\alpha$ if and only if $u \circ \alpha_X : X \longrightarrow Y$ can be completed in $\mathcal{T}$. The latter holds by TR2 for $\mathcal{T}$.

TR3: By definition, $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X) \in \mathcal{T}_\alpha \iff X \xrightarrow{u\circ\alpha_X} Y \xrightarrow{v} Z \xrightarrow{w} T(X) \in \mathcal{T}$. By TR3 for $\mathcal{T}$, the latter holds if and only if $Y \xrightarrow{-v} Z \xrightarrow{-w} T(X) \xrightarrow{-T(u\circ\alpha_X)} T(Y) \in \mathcal{T}$. As α commutes with T, $T(u \circ \alpha_X) = T(u) \circ \alpha_{T(X)}$. So, by remark 7.3, $Y \xrightarrow{-v} Z \xrightarrow{-w} T(X) \xrightarrow{-T(u)} T(Y)$ is in $\mathcal{T}_\alpha$. This proves TR3.

TR5: Suppose we have the following distinguished triangles in $\mathcal{T}_\alpha$:

$X \xrightarrow{f} Y \xrightarrow{h} Z' \xrightarrow{m} T(X)$, $X \xrightarrow{g\circ f} Z \xrightarrow{l} Y' \xrightarrow{n} T(X)$, $Y \xrightarrow{g} Z \xrightarrow{k} X' \xrightarrow{p} T(Y)$.

By TR5 for $\mathcal{T}$, the following commutative diagram exists and the lowest row is a distinguished triangle:

$$\begin{tikzcd}
X \arrow[r, "f\circ\alpha_X"] \arrow[d, equal] & Y \arrow[r, "h"] \arrow[d, "g\circ\alpha_Y"] & Z' \arrow[r, "m"] \arrow[d, dashed, "u"] & T(X) \arrow[d, equal] \\
X \arrow[r, "g\circ\alpha_Y\circ f\circ\alpha_X"] \arrow[d, "f"] & Z \arrow[r, "l"] \arrow[d, equal] & Y' \arrow[r, "n"] \arrow[d, dashed, "v"] & T(X) \arrow[d, "T(f)"] \\
Y \arrow[r, "g\circ\alpha_Y"] \arrow[d, "h"] & Z \arrow[r, "k"] \arrow[d, "l"] & X' \arrow[r, "p"] \arrow[d, equal] & T(Y) \arrow[d, "T(h)"] \\
Z' \arrow[r, dashed, "u"] & Y' \arrow[r, dashed, "v"] & X' \arrow[r, dashed, "w"] & T(Z')
\end{tikzcd}$$

Observe that, by definition of α, $g \circ \alpha_Y = \alpha_Z \circ g$ and $l \circ \alpha_Z = \alpha_{Y'} \circ l$. Thus, $u \circ h = l \circ g \circ \alpha_Y = \alpha_{Y'} \circ l \circ g$. Moreover, by assumption α is a global automorphism, therefore $\alpha_{Y'}^{-1}$ is well-defined.

Thus, the following diagram commutes and the last line is in $\mathcal{T}_\alpha$:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "h"] \arrow[d, "g"] & Z' \arrow[r, "m"] \arrow[d, dashed, "\alpha_{Y'}^{-1}\circ u"] & T(X) \arrow[d, equal] \\
X \arrow[r, "g\circ f"] \arrow[d, "f"] & Z \arrow[r, "l"] \arrow[d, equal] & Y' \arrow[r, "n"] \arrow[d, dashed, "v"] & T(X) \arrow[d, "T(f)"] \\
Y \arrow[r, "g"] \arrow[d, "h"] & Z \arrow[r, "k"] \arrow[d, "l"] & X' \arrow[r, "p"] \arrow[d, equal] & T(Y) \arrow[d, "T(h)"] \\
Z' \arrow[r, dashed, "\alpha_{Y'}^{-1}\circ u"] & Y' \arrow[r, dashed, "v"] & X' \arrow[r, dashed, "w"] & T(Z')
\end{tikzcd}$$

$\square$

Remark 7.5. Observe that α such that $\alpha_A = -id_A$ is a global automorphism in any triangulated category and that $\mathcal{T}_\alpha$ is precisely the class of anti-distinguished triangles. In particular, we just proved that the class of anti-distinguished triangles defines a triangulation as claimed in definition 3.9.

Having now seen that we can associate a triangulation to any global automorphism, we wonder if the triangulation induced by the global automorphism can be different from the initial one and if so when. The following proposition illustrates a case where they are indeed different. Before we state the proposition, we recall some terminology.

Definition 7.6. As a special case of example 4.3, for some commutative ring with unit, we define the categories $Gr(Mod(R))_c$ and $(K_c(Gr(Mod(R))), T)$. Recall that the objects of $Gr(Mod(R))_c$ are chain complexes of R-modules and its morphisms are morphisms of chain complexes as defined in example 4.3. The category $(K_c(Gr(Mod(R))), T)$ is the associated homotopy category as defined in remark 4.13.

Recall from theorem 4.11 that $(K_c(Gr(Mod(R))), T)$ is triangulated and denote by $\mathcal{T}$ its usual triangulation where the distinguished triangles are those isomorphic to the mapping cone triangle. Furthermore, remember that, as before, T shifts a complex by $+1$. For a given chain complex the module at degree $n-1$, A^{n-1} maps to A^n so, as complexes are numbered in increasing order from left to right, T shifts everyone to the left by one.

Proposition 7.7. *Let R be a unitary commutative ring with more than one unit, $b \in R^\times$ a unit different from 1. Suppose there exists an element $r \in R$ such that r is not a zero-divisor and r does not divide $1-b$. Consider the triangulated category $(K_c(Gr(Mod(R))), T)$. Let λ_b be the global automorphism of $K_c(Gr(Mod(R)))$ given by multiplication by b i.e. $\lambda_b(A_\bullet)$ is the chain map given by multiplication by b on every degree. Then, $\mathcal{T}$ and $\mathcal{T}_{\lambda_b}$ are two different triangulations on $K_c(Gr(Mod(R)))$.*

Proof. Suppose for the sake of contradiction that $\mathcal{T}_{\lambda_b} = \mathcal{T}$. We can see R as a chain complex $R_\bullet$ centered at degree 0 and the morphism $R \longrightarrow R$ given by multiplication by r as a map of chain complexes concentrated at degree 0.

$$\begin{tikzcd} \cdots \arrow[r] & 0 \arrow[r] \arrow[d, "r"] & R \arrow[r] \arrow[d, "r"'] & 0 \arrow[r] \arrow[d, "r"'] & \cdots \\ \cdots \arrow[r] & 0 \arrow[r] & R \arrow[r] & 0 \arrow[r] & \cdots \end{tikzcd}$$

We do the same thing for multiplication by b.
Let $Mc(r) = T(R) \oplus R$ be the mapping cone of r. $Mc(r)$ is given by the following chain complex

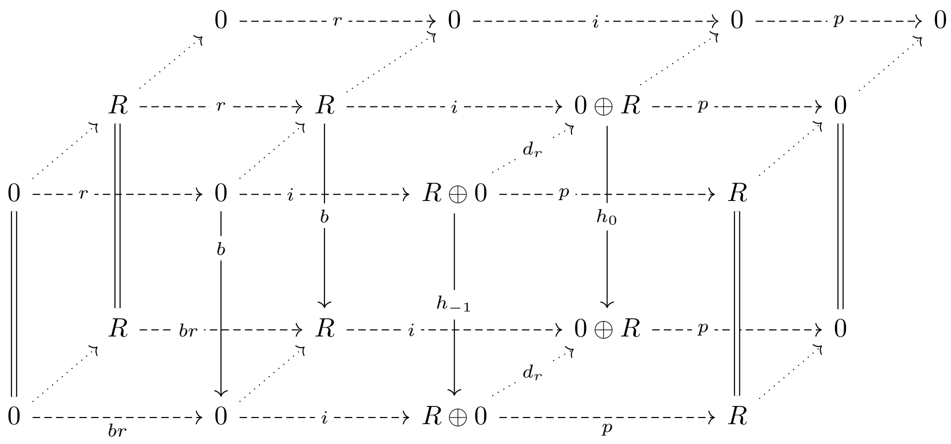
$$\cdots \longrightarrow 0 \longrightarrow R \oplus 0 \xrightarrow{d_r} 0 \oplus R \longrightarrow 0 \longrightarrow \cdots$$

where $0 \oplus R$ is at degree 0, $R \oplus 0$ at degree -1. By definition, its differential d_r is $d_r(x \oplus 0) = 0 \oplus rx$. We have the natural maps $i : R_\bullet \longrightarrow Mc(r)$ given by inclusion in the second component and $p : Mc(r) \longrightarrow T(R_\bullet)$ by projection from the first component.
We get the following diagram:

$$\begin{tikzcd} R_\bullet \arrow[r, "r"] \arrow[d, equal] & R_\bullet \arrow[r, "i"] \arrow[d, "b"'] & Mc(r) \arrow[r, "p"] \arrow[d, dashed, "h"] & T(R_\bullet) \arrow[d, equal] \\ R_\bullet \arrow[r, "br"'] & R_\bullet \arrow[r, "i"'] & Mc(r) \arrow[r, "p"'] & T(R_\bullet) \end{tikzcd}$$

which commutes up to homotopy. By assumption, the two lines are distinguished triangles, the existence of h is given by TR4.

Writing out the chain complexes, we have:



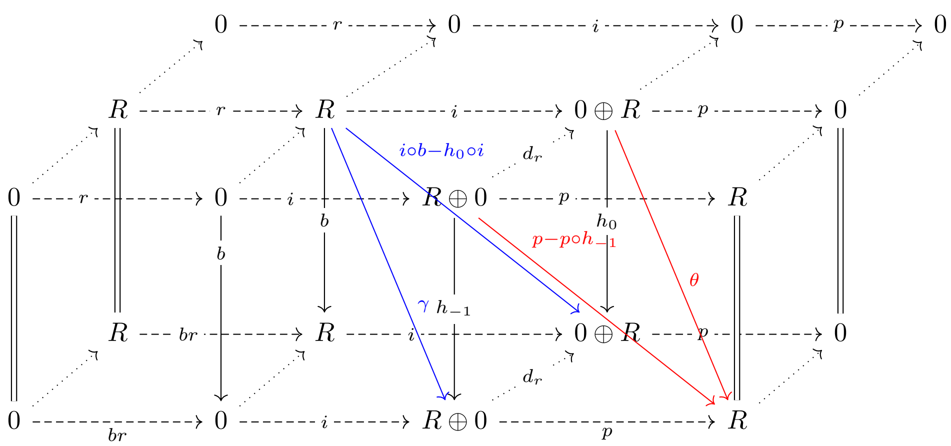
All the maps considered here are R-module homomorphisms, so they are uniquely determined by the image of 1. Since h is a chain map, it has to satisfy $h_0 \circ d_r = d_r \circ h_{-1}$. In particular, we have that $r(h_0(0 \oplus 1) - h_{-1}(1 \oplus 0)) = 0$ because $h_0(r \cdot (0 \oplus 1)) = rh_0(0 \oplus 1) = rh_{-1}(1 \oplus 0)$. Since we assumed that r is not a zero divisor, we deduce that $h_{-1}(1 \oplus 0) = h_0(0 \oplus 1)$ and more generally $h_{-1}(r' \oplus 0) = h_0(0 \oplus r')\ \forall r' \in R$. Denote the projections $\pi_1 : 0 \oplus R \to R$ and $\pi_2 : R \oplus 0 \to R$. With this notation, $\pi_1 \circ h_0 = \pi_2 \circ h_{-1}$.
Since the diagram commutes up to homotopy, we get the existence of a map $\gamma : R \longrightarrow R \oplus 0$ satisfying $i \circ b - h_0 \circ i = d_r \circ \gamma$ i.e we have the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & R \arrow[r] \arrow[dl, "\gamma_0=\gamma"'] \arrow[d, "i\circ b-h_0\circ i"] & 0 \arrow[dl, "\gamma_1=0"] \\
0 \arrow[r] & R\oplus 0 \arrow[r, "d_r"'] & 0\oplus R \arrow[r] & 0
\end{tikzcd}$$

Similarly, we get the existence of a map $\theta : 0 \oplus R \longrightarrow R$ such that $\theta \circ d_r = p - p \circ h_{-1}$ i.e. we have the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & R\oplus 0 \arrow[r, "d_r"] \arrow[dl, "\theta_{-1}=0"'] \arrow[d, "p-p\circ h_1"] & 0\oplus R \arrow[dl, "\theta_0=\theta"] \\
0 \arrow[r] & 0 \arrow[r] & R \arrow[r] & 0
\end{tikzcd}$$

Writing out all the chain complexes, we get the following diagram (γ in blue and θ in red):



Therefore,

$$0 \oplus b = i \circ b(1) = d_r \circ \gamma(1) + h_0 \circ i(1) = 0 \oplus r\gamma(1) + h_0(0 \oplus 1).$$

In particular, $b = r\gamma(1) + \pi_1 \circ h_0(0 \oplus 1)$. Moreover,

$$1 = p(1 \oplus 0) = \theta \circ d_r(1 \oplus 0) + p \circ h_{-1}(1 \oplus 0) = \theta(0 \oplus r) + \pi_2 \circ h_{-1}(1 \oplus 0) = r\theta(0 \oplus 1) + \pi_2 \circ h_{-1}(1 \oplus 0).$$

Remembering $\pi_2 \circ h_{-1}(1 \oplus 0) = \pi_1 \circ h_0(0 \oplus 1)$, we notice that $1 - b = r(\theta(0 \oplus 1) - \gamma(1))$. In other words, $1 - b$ is divisible by r, which contradicts our initial assumption. Thus, $\mathcal{T} \neq \mathcal{T}_{\lambda_b}$. □

Remark 7.8. Consider the category $K_c(Gr(Ab))$. We already noticed that the class of anti-distinguished triangles is given by $\mathcal{T}_{-1}$. Choose r any non-zero element in $\mathbb{Z}$ which does not divide $1 - (-1) = 2$, for example set $r = 3$. Then, proposition 7.7 implies that the two triangulations do not coincide, as stated earlier.

All that is left to see now, is that we can find a ring satisfying these assumptions for infinitely many triangulations. This is shown in the next theorem.

Theorem 7.9. *There exists a category (K, T) with infinitely many distinct triangulations.*

Proof. Let S be a commutative domain with infinitely many units, for example an infinite field like $\mathbb{R}$. Set $R = S[x]$ the polynomial ring in one variable over S and consider $K = K_c(Gr(Mod(R)))$.
$x \in R$ is not a zero-divisor and it doesn't divide $1 - b$ for b any unit different from 1. So, using the notation from proposition 7.7, we can set $r = x$ and we get $\mathcal{T}_b \neq \mathcal{T}$ for all (infinitely many) units in R.
Notice that if $\mathcal{T}_b = \mathcal{T}_c$, we have $\mathcal{T}_{b^{-1}c} = \mathcal{T}$. Indeed, suppose $\mathcal{T}_b = \mathcal{T}_c$ and consider a triangle that is distinguished with respect to $\mathcal{T}$ which we denote $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$.

Then, $X \xrightarrow{ub^{-1}} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$ is distinguished in $\mathcal{T}_b$ which equals $\mathcal{T}_c$ by assumption. By definition of $\mathcal{T}_c$, it follows that the triangle $X \xrightarrow{ub^{-1}c} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$ is distinguished with respect to $\mathcal{T}$. So, by definition of $\mathcal{T}_{b^{-1}c}$, $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X) \in \mathcal{T}_{b^{-1}c}$. Therefore, $\mathcal{T} \subseteq \mathcal{T}_{b^{-1}c}$.
Conversely, suppose the triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$ is distinguished with respect to $\mathcal{T}_{b^{-1}c}$. This is equivalent to the triangle $X \xrightarrow{ub^{-1}c} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$ being in $\mathcal{T}$. Thus, if $X \xrightarrow{ub^{-1}} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$ is distinguished with respect to $\mathcal{T}_c = \mathcal{T}_b$, then $X \xrightarrow{ub^{-1}b=u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$ is distinguished in $\mathcal{T}$ which proves the reverse inclusion. This forces $b = c$ by proposition 7.7. Thus, all triangulations $\mathcal{T}_b$ are distinct. □

Remark 7.10. From the proof, we know that there are $|S|$ different triangulations in $K_c(Gr(Mod(S[x])))$ and that any infinite field works as a choice for S. It follows from the theorem of Löwenhein-Skolem that there exists a field for any given cardinality. Therefore, one can find a triangulated category with κ distinct triangulations for κ any cardinal.

It is an open problem, whether a triangulated category can admit two different triangulations that are not obtained from each other via global isomorphisms.

8 Conclusion

When one first encounters the definition of triangulated category, one might think that the axioms impose rather restrictive conditions on triangulated categories. After studying them more closely, one can still argue that in some sense this is true: in any triangulated category all epimorphisms and monomorphisms split. This has the constraining consequence that an abelian category is triangulated if and only if it is semisimple or that the arrow category of a triangulated category cannot be triangulated. On the other hand, however, the axiomatic turns out rather rich in structure and leaves a certain degree of freedom. The construction of the homotopy category and the derived category give many examples of triangulated categories arising naturally in mathematics, in particular in algebraic topology and homological algebra. Furthermore, as we have seen in the last section, for any given infinite cardinal, we can find a category with that many distinct triangulations. This is a result one does not suspect when confronted for the first time with the definition.

An important feature of triangulated categories to keep in mind is the fact that they are closely related to exact sequences. We have seen in proposition 3.13 that the composition of two adjacent morphisms in a distinguished triangle gives zero, just like in exact sequences. Moreover, if an abelian category is triangulated the distinguished triangles will necessarily be given by exact sequences. Finally, as was shown in theorem 5.51, every short exact sequence in the initial category corresponds to a ”standard” distinguished triangle in the derived category.

Of course, triangulated category theory doesn't end here. The result presented in this paper are just the foundation to tensor triangular geometry (see [3]) or the proof of the Brown representation theorem which essentially states that any cohomological functor from a triangulated category to the category of $\mathbb{Z}$-modules is representable (see [15] or [10]). As mentioned in the introduction, the study of derived categories is also a relevant research area and its applications extend to many different subjects. Finally, one can study stable infinity categories (see [12] for an introduction) or derivators as proposed by Grothendieck (see [6] or [7]) to generalise and improve the properties of triangulated categories.

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